ON THE FUNDAMENTAL GROUP OF THE SURFACE PARAMETRIZING CUBOIDS

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Abstract. We compute the fundamental group of the complex and real points of the surface parametrizing cuboids and of their respective minimal resolutions of singularities. We also compute the fundamental group of the analogous sets for the surface parametrizing face cuboids.

1. Introduction

In this note we study the fundamental groups of the surface parametrizing cuboids (also called box variety in [4]) and of its resolution.

A cuboid is a hexahedron, characterized by seven lengths. These come from the three edges A,B,C, the three face diagonals X,Y, Z, and the diagonal U, related by the following four equations:

$$\begin{aligned} A^2+B^2-Z^2&=0\\ B^2+C^2-X^2&=0\\ C^2+A^2-Y^2&=0\\ A^2+X^2-U^2&=0. \end{aligned}$$

A *perfect* cuboid is defined to be a cuboid such that the seven lengths are in $\mathbb{Z}$, and the non-existence of such a cuboid is an old and sought-after conjecture whose origin dates back to Euler.

The surface parametrizing cuboids, defined by Van Lujik [12], is the surface Υ in $\mathbb{P}^6$ defined by the four equations above. Van Lujik shows that Υ is a complete intersection and has 48 complex singular points which are ordinary double points. He also constructs its desingularization $\widetilde{\Upsilon}$. An alternative moduli interpretation involving modular curves, can be found in [4].

Our main result goes as follows.

Theorem 1.1. *Let the symbol $\sim$ denote the minimal resolution of singularities. We have:*

1. $\pi_1(\Upsilon_{\mathbb{C}})=\pi_1(\widetilde{\Upsilon}_{\mathbb{C}})=0$, *where $\Upsilon_{\mathbb{C}}$ is the set of complex points of Υ, viewed as a complex projective variety.*
2. $\pi_1(\Upsilon_{\mathbb{R}})=\pi_1(N_{48-k})*\mathbb{F}_{24}$ *and $\pi_1(\widetilde{\Upsilon}_{\mathbb{R}})=\pi_1(N_k)$ where $\Upsilon_{\mathbb{R}}$ is the set of real points of Υ, viewed as a real projective variety, $\mathbb{F}_{24}$ is the free group of 24 generators, N_r is the connected sum of r copies of $\mathbb{P}^2_{\mathbb{R}}$, and k a positive integer described in §3.*

We prove part (1) of Theorem 1.1 in §2 and part (2) in §3. We also describe the fundamental group of the real and complex points of the surface of face cuboids of Υ in §4, i.e., $\Phi = \Upsilon/\sim$, where $\sim$ here denotes the relation defined by $Z \sim -Z$.

Our motivation for these results comes from the application of the Kim-Chabauty method [9], which is a method to study the integral points of hyperbolic non-singular algebraic curves, and a major tool in its is the crystalline realization of the unipotent fundamental group, which can be computed via its topological fundamental group over the complex numbers. By Faltings' theorem there exists only finitely many integral points in such a curve and the Chabauty-Kim method provides a way to determine these points explicitly. Although the Kim-Chabauty method has been mainly applied to curves, an example of an extension of the method to surfaces is given in [2]. Here the method would be applied to $\widetilde{\Upsilon}$ minus a divisor.

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2. The fundamental group of the surface parametrizing complex cuboids

Consider $R = \mathbb{C}[A, B, C, X, Y, Z, U]$ and let H be the divisor defined by an equation of the following form (for some $\alpha, \beta, \gamma \in \mathbb{C}$)

$$\alpha A^2 + \beta B^2 + \gamma C^2 = \delta.$$

Lemma 2.1. *The following statements hold.*

(1) *The ideal $I \subset R$ defining $\Upsilon_{\mathbb{C}} \cap H$ is a prime ideal;*
(2) *consider the open affines given by equating one the variables to 1, that is, $A = 1$, $B = 1$, $C = 1$, $X = 1$, $Y = 1$, $U = 1$, $Z = 1$. Then the ideal I remains prime under these respective restrictions.*
(3) *Υ is path-connected and so is its intersection with H.*
(4) *The minimal resolution of singularities of $\Upsilon_{\mathbb{C}}$, namely $\widetilde{\Upsilon}_{\mathbb{C}}$, is connected.*

Proof. The first two parts are variants of [12, Lemma 3.2.1]. Part 3 follows from part 1 and from the fact that irreducible complex varieties are path connected in the Euclidean topology. Part 4 follows from the fact that the blow up of an irreducible variety is irreducible, as the fibers are connected by Zariski's main theorem (see, for instance, [6, Section 12.6]). $\square$

We now recall the Lefschetz hyperplane theorem, which we use in our proof of the proof of Theorem 2.3.

Theorem 2.2 (Lefschetz's Hyperplane)**.** *Let H be an ample divisor on a manifold X of dimension n, and let $i \colon H \hookrightarrow X$ be its inclusion. Then, for $j < n-1$*

$$\pi_j(i) \colon \pi_j(H) \simeq \pi_j(X)$$

and π_{n-1} is a surjection.

[1]In turn, Ishai Dan-Cohen wishes to thank Ambrus Pál for introducing *him* to this problem. To the best of our knowledge, the idea to approach the arithmetic of cuboids via fundamental group techniques originates from him.

Proof. See [13, 1.1]. □

The following result's proof follows the idea of the first part of [3, Theorem 2.1].

Theorem 2.3. *The fundamental group of $\Upsilon_{\mathbb{C}}$ is trivial. Moreover, $\widetilde{\Upsilon_{\mathbb{C}}}$ has a trivial fundamental group as well.*

Proof. Let H be a non-trivial hyperplane that intersects Υ non-trivially and generically. We set $W := \Upsilon_{\mathbb{C}} \cap H$ and $U = \Upsilon_{\mathbb{C}} - W$, and let N be a tubular neighborhood of H in $\Upsilon_{\mathbb{C}}$. We exploit Van Kampen's theorem to prove simply connectedness of $\Upsilon_{\mathbb{C}}$. Consider the covering (U, W) of $\Upsilon_{\mathbb{C}}$ and note that U and W are open and path-connected. Suppose that $U \cap N$ is path-connected and non trivial, and fix a base point $x_0 \in U \cap N$ which we will omit. Let the following morphisms $j_1 \colon \pi_1(U) \to \pi_1(\Upsilon_{\mathbb{C}})$ and $j_2 \colon \pi_1(N) \to \pi_1(\Upsilon_{\mathbb{C}})$ be induced by the inclusion maps.

By Lemma 2.1, we have that $\Upsilon_{\mathbb{C}}$ is path-connected, and so j_1, j_2 form the following commutative pushout diagram:

$$\begin{tikzcd}
\pi_1(U \cap N) \arrow[r, "i_1"] \arrow[d, "i_2"] & \pi_1(U) \arrow[d, "j_1"] \\
\pi_1(N) \arrow[r, "j_2"] & \pi(\Upsilon_{\mathbb{C}}) \simeq \pi_1(N) *_{\pi_1(N \cap U)} \pi_1(U)
\end{tikzcd}$$

As in [3, p.324] we have that U is homeomorphic to a singular fiber in a deformation of a singularity. Therefore by [8, Corollary 5.9] we deduce that U is a bouquet of 2-spheres, i.e., that U is a wedge sum of S^2's. Thus $\pi_1(S^2) = 0$. This immediately gives us that $\pi_1(U)$ is zero. On the other hand, it is now enough to apply Lefschetz hyperplane theorem 2.2 to notice that H has the same fundamental group as $\mathbb{P}^6$, which is trivial. Since N is a tubular neighborhood of H, they has the same fundamental. Lastly, we have that both U and N have trivial fundamental group. Hence their free product is trivial, and so we can conclude that $\pi_1(\Upsilon_{\mathbb{C}}) = 0$.

Finally, consider case of $\widetilde{\Upsilon_{\mathbb{C}}}$. The tubular neighborhoods of a singular point and of the corresponding exceptional divisor have respectively the homothopy type of a point and of $\mathbb{P}^1$. Hence they are both simply connected and in the Zariski–Van Kampen diagram nothing changes, so they have the same fundamental group. □

Remark 2.4. This immediately verifies what was conjectured in [12, Bluff 1, p. 31]. Indeed, we have

$$0 = \pi_1(\widetilde{\Upsilon_{\mathbb{C}}}) \twoheadrightarrow H_1(\widetilde{\Upsilon_{\mathbb{C}}}) \simeq H^1(\widetilde{\Upsilon_{\mathbb{C}}}).$$

For the Hodge diamond of $\widetilde{\Upsilon_{\mathbb{C}}}$, see [11, p.4].

Remark 2.5. Note that Van Luijk [12, Cor. 3.3.34] proves using Noether's formula form Euler characteristic of surfaces that the (topological) Euler characteristic of $\widetilde{\Upsilon_{\mathbb{C}}}$ is 80.

3. The fundamental group of the surface parametrizing real cuboids

We begin with a few observations about $\Upsilon_{\mathbb{R}}$. First, as in Lemma 2.1, since $\Upsilon_{\mathbb{R}}$ is an irreducible compact complete intersection of dimension 2, then $\Upsilon_{\mathbb{R}}$ is path connected.

Second, from the proof of in [12, Corollary 3.2.3], over the real numbers, $\Upsilon_{\mathbb{R}}$ has 24 singularities which are ordinary double points.

Third, by looking at the defining equations of $\Upsilon_{\mathbb{R}}$, (viewed as a projective variety) over the real numbers we must have that $U = 1$, as otherwise, all variables must equal to zero which is obviously impossible in the projective case. Thus we will view $\Upsilon_{\mathbb{R}}$ as a compact affine variety, as we replaced its last defining equation by $A^2 + B^2 + C^2 = 1$. From this we can conclude that $\Upsilon_{\mathbb{R}}$ is compact, as it is contained in $\mathrm{Ball}_1(0)$.

We will use these observations to compute the fundamental group of $\Upsilon_{\mathbb{R}}$ using a few key results.

Proposition 3.1. *There exists a compact differential surface M such that*

$$\pi_1(\Upsilon_{\mathbb{R}}) \cong \pi_1(M) * F_{24}$$

where F_{24} is the free group on 24 generators. In particular, $\Upsilon_{\mathbb{R}}$ is not simply connected.

Proof. We will construct M in the following way. Let $p \in \Upsilon_{\mathbb{R}}$ be an ordinary double point. By Morse's lemma (see [7, Theorem 2.46]), there exists a neighborhood $p \in U \subset \Upsilon$ such that U is homeomorphic to

$$X = V(x^2 + y^2 - z^2) \cap \mathrm{Ball}_1(0) \subset \mathbb{R}^3.$$

Therefore, $U \setminus \{p\}$ is homeomorphic to an open cone from whom we have removed the origin, which itself is homeomorphic to the intersection of a ball with two disjoint cylinders. Therefore, by gluing two disks on either sides of $\Upsilon_{\mathbb{R}} \setminus \{p\}$, we have replaced p be a smooth point, and by performing this to every singular point of $\Upsilon_{\mathbb{R}}$, we constructed a compact differential surface M. Now, we can view $\Upsilon_{\mathbb{R}}$ as M in which we identified $2 \cdot 24 = 48$ points. Yet, identifying a pair of points on a topological surface is the same as attaching a closed CW 1-cell to it. Therefore, $\Upsilon_{\mathbb{R}}$ is homeomorphic to the wedge product of M with 24 circles, and so using Van-Kampen's Theorem, we conclude that

$$\pi_1(\Upsilon_{\mathbb{R}}) = \pi_1(M) * \pi_1(\bigvee^{24} S_1) = \pi_1(M) * F_{24}.$$

□

We recall the classification theorem for differential compact surfaces, which we will use extensively throughout this section. For more details about this result, see, for example, [5].

Proposition 3.2 (Classification of differential compact surfaces)**.** *Let S be a compact differential surface. Then S is homeomorphic to either the sphere S^2, a connected sum of g tori Σ_g, or the connected sum of k projective planes N_k. In particular we have that one of the following is true:*

(1) $\pi_1(S) = 1$,
(2) $\pi_1(S) = \langle a_1, b_1, \ldots, a_g, b_g \colon [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$,
(3) $\pi_1(S) = \langle c_1, \ldots, c_k \colon c_1^2 \cdots c_k^2 = 1 \rangle$.

Proposition 3.3. *Let $\widetilde{\Upsilon}_{\mathbb{R}}$ be the minimal resolution of singularities of $\Upsilon_{\mathbb{R}}$ together with the resolution map $\pi\colon \widetilde{\Upsilon}_{\mathbb{R}} \to \Upsilon_{\mathbb{R}}$. Then $\widetilde{\Upsilon}_{\mathbb{R}}$ is homeomorphic to the connected sum of M with Σ_{24}.*

Proof. Since the only singularities of $\Upsilon_{\mathbb{R}}$ are ordinary double points, then $\widetilde{\Upsilon}_{\mathbb{R}}$ is constructed by blowing up each singularity of $\Upsilon_{\mathbb{R}}$ once. We also note that $\widetilde{\Upsilon}_{\mathbb{R}}$ is path connected since Υ is irreducible, and therefore so is $\widetilde{\Upsilon}_{\mathbb{R}}$. Let $p \in \Upsilon_{\mathbb{R}}$ be an ordinary double point, and so from Morse's lemma there exists a neighborhood $p \in U \subset \Upsilon_{\mathbb{R}}$ such that U is homeomorphic to

$$X = V(x^2 + y^2 - z^2) \cap \mathrm{Ball}_1(0) \subset \mathbb{R}^3.$$

Now, the blow up $V(x^2+y^2-z^2)$ at the origin is homeomorphic to the $\mathbb{R}\times\mathbb{P}^1_{\mathbb{R}}$ as the exceptional divisor is isomorphic to $\mathbb{P}^1_{\mathbb{R}}$, and blowing up $\Upsilon_{\mathbb{R}}$ at p is a homeomorphism outside U. But locally outside p we have that $M \setminus \{p\}$ is homeomorphic to $\widetilde{\Upsilon}_{\mathbb{R}} \setminus \pi^{-1}(p)$. Thus, topologically, the blow up of $\Upsilon_{\mathbb{R}}$ at p replaces the CW $1-$cell which connects the two points corresponding to x in M by a copy of $\mathbb{R} \times \mathbb{P}^1_{\mathbb{R}} = \mathbb{R} \times S^1$. Therefore, it corresponds to the connect sum of M with a torus. By preforming this over all 24 singular points, the result follows. □

Proposition 3.4. *$\widetilde{\Upsilon}_{\mathbb{R}}$ is non orientable.*

Proof. Since $\widetilde{\Upsilon}_{\mathbb{R}}$ is the resolution of $\Upsilon_{\mathbb{R}}$, it is a compact differentiable surface. In addition, we can write the resolution as a series of blow ups at its ordinary double points $\Upsilon_N \to \Upsilon_{N-1} \to \cdots \to \Upsilon_1 \to \Upsilon_0 = \Upsilon_{\mathbb{R}}$. We will prove that for every i we have that $H^2(\Upsilon_i, \mathbb{Z})$ is not torsion free, and so the result follows from the cohomological description of orientability. Let $x_i \in \Upsilon_i$ be the point we blow up under the map $\pi_i\colon \Upsilon_{i+1} \to \Upsilon_i$ and let $E_{i+1} = \pi^{-1}(x_i)$. Then we have a long exact sequence

$$\cdots \to H^1(E_{i+1}, \mathbb{Z}) \to H^2(\Upsilon_i, \mathbb{Z}) \to H^2(\Upsilon_{i+1}, \mathbb{Z}) \oplus H^2(\{x_i\}, \mathbb{Z}) \to H^2(E_{i+1}, \mathbb{Z}) \to \cdots.$$

Yet we have that $H^2(\{x_i\}, \mathbb{Z}) = 0$ and that E_{i+1} is homeomorphic to $\mathbb{P}^1_{\mathbb{R}}$, and thus $H^2(E_{i+1}, \mathbb{Z}) = 0$ and $H^1(E_{i+1}, \mathbb{Z}) = \mathbb{Z}$. Thus the long exact sequence becomes

$$\mathbb{Z} \to H^2(\Upsilon_i, \mathbb{Z}) \to H^2(\Upsilon_{i+1}, \mathbb{Z}) \to 0.$$

Therefore, since the map $H^2(\Upsilon_i, \mathbb{Z}) \to H^2(\Upsilon_{i+1}, \mathbb{Z})$ is surjective then if Υ_i is non orientable then so is Υ_{i+1}, and if Υ_i is orientable we have that $H^2(\Upsilon_i, \mathbb{Z}) = \mathbb{Z}$ and thus $H^2(\Upsilon_{i+1}, \mathbb{Z})$ is the non zero cokernel of a map $\mathbb{Z} \to \mathbb{Z}$, which can never be torsion free. Therefore Υ_{i+1} is never orientable for every i and the result follows. □

Remark 3.5. By the classification theorem we have that $\widetilde{\Upsilon}_{\mathbb{R}}$ is homeomorphic to N_k for some k.

Proposition 3.6. *M is non-orientable and homeomorphic to N_{k-48}.*

Proof. Since $\widetilde{\Upsilon}_{\mathbb{R}}$ is non orientable and homeomorphic to the connected sum of M with Σ_{24}, then if M would have been orientable, then so would $\widetilde{\Upsilon}_{\mathbb{R}}$. Thus M is homeomorphic to N_r for some r. Thus the connected sum of N_r and Σ_{24} is homeomorphic N_{48+r}, but is homeomorphic to $\widetilde{\Upsilon}_{\mathbb{R}}$, which itself is homeomorphic to N_k. Thus $k = 48 + r$ and the result follows. □

Proposition 3.7. *We have $k = 26 - \chi(\Upsilon_{\mathbb{R}})$ where $\chi(\Upsilon_{\mathbb{R}})$ is the topological Euler characteristic of $\Upsilon_{\mathbb{R}}$.*

Proof. Since $\Upsilon_{\mathbb{R}}$ is constructed by attaching 24 CW 1-cells to M, we get that $\chi(\Upsilon_{\mathbb{R}}) = \chi(M) - 24$. But since M is homeomoprhic to N_{k-48}, we have that $\chi(M) = 2-(k-48) = 50-k$. Thus $\chi(\Upsilon_{\mathbb{R}}) = (50-k)-24 = 26-k$ which gives us that $k = 26 - \chi(\Upsilon_{\mathbb{R}})$. □

Now, we recall a result by Bruce from [1].

Proposition 3.8 (Proposition 7 in [1]). *Let $f_1, \dots, f_r \colon \mathbb{R}^n \to \mathbb{R}$ be polynomials of degree $\leq d$ and suppose $X = V(f_1, \dots, f_r)$ is compact in the Euclidean topology of $\mathbb{R}^n$. Then*

$$\chi(X) = \frac{(-1)^n - \mu(H)}{2}$$

where $\chi(X)$ is the topological Euler characteristic of X and $H \colon \mathbb{R}^{n+1} \to \mathbb{R}$ is defined to be

$$H(\underline{x}, y) = (\sum_{i=1}^{r} y^{d+1} f(\frac{x_1}{y}, \dots, \frac{x_n}{y})) - y^{2d+4} - x_1^{2d+4} - \cdots x_n^{2d+4},$$

and by $\mu(H)$ we mean the real milnor number of H at the origin, i.e. the dimension of $\mathbb{R}[[x_1,\dots,x_n]]/\langle \frac{\partial H}{\partial x_1}, \dots, \frac{\partial H}{\partial x_n} \rangle$ as a real vector space, where $\mathbb{R}[[x_1, \dots, x_n]]$ is the ring of real analytic functions in variables $x_1, \dots, x_n$.

Now, in order to compute $\chi(\Upsilon_{\mathbb{R}})$ we will use Bruce's formula, as described above. In our case we have that Υ is defined by the equations

$$\begin{aligned} A^2 + B^2 - Z^2 &= 0 \\ B^2 + C^2 - X^2 &= 0 \\ C^2 + A^2 - Y^2 &= 0 \\ A^2 + B^2 + C^2 - 1 &= 0. \end{aligned}$$

Therefore, H will be a polynomial in 7 variables, namely A, B, C, X, Y, Z, D which will be $H(A,B,C,X,Y,Z,D) = D^2((A^2+B^2-Z^2)^2 + (B^2+C^2-X^2)^2 + (C^2 + A^2-Y^2)^2 + (A^2+B^2+C^2-D^2)^2) - A^8 - B^8 - C^8 - X^8 - Y^8 - Z^8 - D^8$, and $\chi(\Upsilon_{\mathbb{R}})$ can be computed from the value of $\mu(H)$.

We thus immediately obtain the following Corollary.

Corollary 3.9. *The fundamental group $\pi_1(\Upsilon_{\mathbb{R}})$ is isomorphic to the free product of F_{24} and $\pi_1(N_{k-48})$.*

4. The fundamental group of the surface parametrizing face cuboids

Van Lujik also considers the following surface. Let $\equiv$ denote the relation defined by $Z \equiv -Z$, and let us consider the surface $\Phi := \Upsilon/\equiv$, which is described in $\mathbb{P}^5_{\mathbb{Q}}$ by

$$\begin{aligned} A^2 + C^2 - Y^2 &= 0 \\ B^2 + C^2 - X^2 &= 0 \\ A^2 + X^2 - U^2 &= 0. \end{aligned}$$

As showed in [12, 4.1 p.51], this surface is a complete intersection with 16 isolated singularities. Moreover its resolution $\widetilde{\Phi}$ is shown to be a K3 surface which is isomorphic to the Kummer surface of the product of the two following elliptic curves with complex multiplication. Consider

$$E: \quad y^2z = x^3 - 4xz^2$$
$$E': \quad y^2z = x^3 + xz^2$$

and the automorphism ι of $E\times E'$ sending (P,Q) to $(-P,-Q)$, where the symbol $-$ refers to the inverse in the group law of the elliptic curve. Then we have

$$\Phi \simeq (E\times E')/\langle\iota\rangle.$$

For more details, see [12, p.53]. We prove the following result.

Proposition 4.1. *We have* $\pi_1(\Phi(\mathbb{C})) = (\mathbb{Z}/2\mathbb{Z})\ltimes\mathbb{Z}^4$ *and* $\pi_1(\widetilde{\Phi}(\mathbb{C})) = 0$.

Proof. Since E and E' are complex elliptic curves, it is well known that both are homeomorphic to $\mathbb{C}/\mathbb{Z}^2$. Therefore, $\Phi(\mathbb{C})$ is homeomorphic to $(\mathbb{C}/\mathbb{Z}^2)^2/(\mathbb{Z}/2\mathbb{Z}) \simeq (\mathbb{C}^2/\mathbb{Z}^4)/(\mathbb{Z}/2\mathbb{Z})$, where $\mathbb{Z}/2\mathbb{Z}$ acts on $(\mathbb{C}/\mathbb{Z}^2)^2$ by $(-1).(x_1,x_2) = (-x_1,-x_2)$. Thus we have a composition of two coverings

$$\mathbb{C}^2 \twoheadrightarrow \mathbb{C}^2/\mathbb{Z}^4 \twoheadrightarrow (\mathbb{C}^2/\mathbb{Z}^4)/(\mathbb{Z}/2\mathbb{Z}) \simeq \Phi(\mathbb{C}),$$

where the second covering above has finite fibers. Therefore, The map

$$p\colon \mathbb{C}^2 \twoheadrightarrow (\mathbb{C}^2/\mathbb{Z}^4)/(\mathbb{Z}/2\mathbb{Z})$$

obtained above is also a covering. Since $\mathbb{C}^2$ is simply connected, this map the universal covering of Φ. Thus the fundamental group of Φ is the group of automorphisms of this covering map. By an argument of connectedness, we can check that these automorphisms are the maps $\mathbb{C}^2\to\mathbb{C}^2$ of the following form:

$$\imath_{(\epsilon,n_1,n_2,n_3,n_4)} : (z,z') \mapsto (\epsilon z + n_1 + in_2, \epsilon z' + n_3 + in_4)$$

where $(\epsilon,n_1,n_2,n_3,n_4)\in(\mathbb{Z}/2\mathbb{Z})\times\mathbb{Z}^4$.

Moreover, the composition of two such maps is given as follows :

$$\imath_{(\epsilon',n_1',n_2',n_3',n_4')} \circ \imath_{(\epsilon,n_1,n_2,n_3,n_4)} = \imath_{(\epsilon'\epsilon,\epsilon' n_1+n_1',\epsilon' n_2+n_2',\epsilon' n_3+n_3',\epsilon' n_4+n_4')}.$$

Therefore, the group of automorphisms is isomorphic to the semi-direct produt $(\mathbb{Z}/2\mathbb{Z})\ltimes\mathbb{Z}^4$ defined as

$$(\epsilon',n_1',n_2',n_3',n_4')\times(\epsilon,n_1,n_2,n_3,n_4) = (\epsilon'\epsilon,\epsilon' n_1+n_1',\epsilon' n_2+n_2',\epsilon' n_3+n_3',\epsilon' n_4+n_4')$$

The triviality of the fundamental group of the resolution $\widetilde{\Phi}$ comes from [10], where Spanier shows that the resolution of $(S^1\times S^1)^n/(\mathbb{Z}/2\mathbb{Z})$ is simply connected. □

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