A PROFINITE THEORY OF DISTRIBUTIONS: TOWARDS THE BATEMAN-HORN CONJECTURE

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1. Introduction

Let $\mathcal{P}$ denote the set of prime numbers. The classical Bateman-Horn conjecture (see [2], [5]) states what follows.

Conjecture 1.1 (Bateman-Horn). *Let $f_1, \ldots, f_k \in \mathbb{Z}[x]$ be distinct irreducible polynomials with positive leading coefficients and f their product. Let*

$$\omega_f(p) := |\{[a] \in \mathbb{Z}/p\mathbb{Z} \text{ such that } f(a) \equiv 0 \text{ mod } (p)\}|.$$

By denoting

$$C(f) := \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right)^{-1} \left(1 - \frac{\omega_f(p)}{p}\right) \tag{1}$$

it follows that

$$\left|\varphi^{-1}(\mathcal{P}^k) \cap [0, x]\right| \sim \frac{C(f)}{\prod_{i=1}^k \deg(f_i)} \int_2^x \frac{dt}{(\log(t))^k}$$

where $\varphi \colon \mathbb{Z} \to \mathbb{Z}^k$ is the map $a \mapsto (f_1(a), \ldots, f_k(a))$ and $\sim$ denotes the same asymptotic behaviour as x tends to $+\infty$.

Several authors have recently tackled this conjecture, either with variations in the function fields setting as [9], [3] and [11], or in a more probabilistic manner as [7] and [21]. In a lower level of generality other relevant results can be found, for instance, in [4], [6] and [16].

In this paper we propose a *profinite* analogue of Conjecture 1.1 by extending the methods developed in [10], i.e., we exploit measure (or rather distribution) theoretic arguments. For a different categorical approach still involving measures, see [1].

Let $\widehat{\mathbb{Z}}$ be the profinite completion of $\mathbb{Z}$. Dirichlet's theorem on primes in arithmetic progressions implies that (and is actually equivalent to, see [14]) the closure of $\mathcal{P}$ in $\widehat{\mathbb{Z}}$ is

$$\widehat{\mathcal{P}} = \mathcal{P} \sqcup \widehat{\mathbb{Z}}^*. \tag{2}$$

Let $\hat{\pi}_n \colon \widehat{\mathbb{Z}} \to \mathbb{Z}/n\mathbb{Z}$ denote the canonical projection and consider the counting measures

$$\mu_{n,\varphi^{-1}(\mathcal{P}^k)} := \frac{1}{|\hat{\pi}_n(\varphi^{-1}(\mathcal{P}^k))|} \sum_{x \in \hat{\pi}_n(\varphi^{-1}(\mathcal{P}^k))} \delta_x$$

on $\mathbb{Z}/n\mathbb{Z}$, as n varies in the positive integers, and for δ_x the Dirac delta at the point x. In §2.2.3 we will define liftings $\tilde{\mu}_{n,\varphi^{-1}(\mathcal{P}^k)}$ of $\mu_{n,\varphi^{-1}(\mathcal{P}^k)}$ to the algebra of measures on $\widehat{\mathbb{Z}}$. Our profinite analogue of Conjecture 1.1 consists of the following statement.

Conjecture 1.2 (profinite Bateman-Horn). *Under the same hypotheses as in Conjecture 1.1 one has*

$$\lim_{n \to 0} \tilde{\mu}_{n,\varphi^{-1}(\mathcal{P}^k)} = \mu_{\varphi^{-1}((\widehat{\mathbb{Z}}^*)^k)}. \tag{3}$$

Note that the existence of the limit on the left-hand side of (3) is far from obvious and is actually part of our conjecture: indeed, we do not know how to prove it. As for the right-hand side of (3), it will be constructed in Proposition 3.16, as the procounting measure attached to $\varphi^{-1}((\widehat{\mathbb{Z}}^*)^k)$ (i.e., the limit of the counting measures on the images of $\varphi^{-1}((\widehat{\mathbb{Z}}^*)^k)$ modulo n).

What is the relation between Conjectures 1.1 and 1.2? We are still unable to prove any implication from one to the other, in any direction. However, both claims fit in a generic philosophy of "irreducible polynomials take lots of prime values, as many as allowed by local conditions": in particular, both of them imply the Schinzel conjecture (as we are going to prove in §3.3.3). Also, we shall show that the local factors of (1) appear as well in the construction of $\mu_{\varphi^{-1}((\widehat{\mathbb{Z}}^*)^k)}$.

We moreover underline that, while in the classical cases one considers $\mathbb{Z}$ embedded into $\mathbb{R}$, heavily relying on the Euclidean topology, our profinite approach is indeed completely independent of any archimedean structure, therefore naturally extending to global fields (as we do in Conjecture 3.18). In particular, in view of the literature cited above, the function field case of conjecture 1.2 can be particularly interesting.

Finally, our conjecture may resemble an equidistribution statement: note that, while usually equidistribution deals with a sequence of sets (i.e., indexed by natural numbers), in our case the set of indexes varies in the *super*natural numbers.

We now briefly describe the structure of the article.

In Section 2, we develop in broad generality a theory of distributions on profinite sets. Despite most of the ideas exploited there are not novel in their essence, to the best of the authors' knowledge they were not treated elsewhere in the terms we propose, which are indeed foundational to the formulation of Conjecture 1.2. Another approach to distribution on profinite sets can be found, for instance, in [8] and its citation orbit; we point out that the intersection with our techniques and result is almost empty.

The crucial new concept of this section consists of the *procounting* distributions, introduced in 2.21, followed by their studies on products and on *close pairs* (see 2.37). It seems plausible to us that it is possible to exploit these techniques far beyond the goals of this paper. For instance, since field extensions are determined by their profinite Galois groups, we expect that our theory of profinite distribution may apply to count certain extension of $\mathbb{Q}$.

In Section 3, we extend [10] to problems where its methods do not hold, i.e., to some sets of (Haar) measure zero. In particular, we focus on *openly Eulerian* sets (see Section 3.2 for this definition). The theory of Section 2 allows to associate a measure with these sets, so that we can quantify how small one of these zero-measure sets is. Moreover, we push our technique beyond openly Eulerian sets, to a wider family of sets which are close to be openly Eulerian, in the sense that they "differ" by a very small set. The crucial such set is $\mathcal{P}$, that leads us to Conjecture 1.2. We conclude this section by showing the coherence of our conjecture in a few classical cases, namely, Dirichlet's theorem, the Twin Primes conjecture and Landau's conjecture. Furthermore we show how the profinite Bateman-Horn conjecture implies Schinzel's hypothesis.

We conclude our paper with an appendix containing a numerical experiment for the Twin Primes conjecture case, i.e., $k = 1$ and $\varphi(x) = x + 2$, which gives us an empirical reason to be optimistic.

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2. Distributions on Profinite Sets

In the following, by topological group we mean a group G endowed with a topological structure of Hausdorff space such that the group operation is a continuous map from $G \times G$ to G. A topological ring R is defined similarly, with the request that addition and multiplication are continuous. If R is a topological ring, unless otherwise stated, we shall assume that all R-modules are topological groups and the map $R \times M \to M$ expressing the ring action of R on M is continuous. Also, unless otherwise stated, we shall assume that all R-module morphisms are continuous.

For any topological spaces A, B, we write $\mathcal{C}(A, B)$ to denote the set of continuous functions from A to B. The characteristic function of a set S will be denoted by $\mathbf{1}_S$, simplifying it to $\mathbf{1}_x$ when $S = \{x\}$ is a singleton. (By abuse of notation, we will use the same symbol $\mathbf{1}_S$ independently of the unitary ring the characteristic function takes values in.)

2.1. **Distributions on profinite sets.** Throughout Section 2, we fix a profinite set

$$X := \varprojlim_{\alpha\in\mathcal{J}} X_\alpha \tag{4}$$

Here $\{X_\alpha\}_{\alpha\in\mathcal{J}}$ is an inverse system of *finite* sets, labeled by α ranging in some arbitrary directed set of indexes $\mathcal{J}$, with maps $\pi_\alpha^\beta\colon X_\beta \to X_\alpha$ for all $\beta \geqslant \alpha$ in $\mathcal{J}$. Without loss of generality, we can (and will) assume that the maps π_α^β are all surjective. Let $\pi_\alpha\colon X \to X_\alpha$ denote the natural projection.

For simplicity, we also postulate that $\mathcal{J}$ contains a totally ordered countable cofinal subset.

2.1.1. *The profinite topology.* In this paper finite sets are always assumed to have the discrete topology. Then the inverse limit topology on X (that is, the coarsest topology such that all the maps π_α are continuous) makes it a profinite topological space

Lemma 2.1. *If S is a subset of X, its closure is*

$$\bigcap_{\alpha\in\mathcal{J}_0} \pi_\alpha^{-1}(\pi_\alpha(S))\,,$$

where $\mathcal{J}_0$ is any cofinal subset of $\mathcal{J}$.

Proof. Assume x is a point of X outside the closure of S. Then there is a neighbourhood U of x such that $U \cap S = \emptyset$. One inclusion follows observing that, without loss of generality, we can take $U = \pi_\alpha^{-1}(\pi_\alpha(x))$ for some $\alpha \in \mathcal{J}_0$. The opposite inclusion is obvious. □

In the following, we shall say that $S \subseteq X$ is α-*saturated* if $S = \pi_\alpha^{-1}(\pi_\alpha(S))$.

2.1.2. *Continuous and locally constant functions.* Let R be a (commutative and unitary) topological ring: then $\mathcal{C}(X,R)$ becomes a topological R-module with the uniform convergence topology.

In order to simplify some arguments, we shall always assume that the topology on R is induced by an absolute value $|\cdot|_R$, so that $\mathcal{C}(X,R)$ is endowed with the supremum norm $\|\cdot\|_\infty$.

Remark 2.2. For further reference, we note that $\mathcal{C}(X_\alpha,R)$ is a free R-module with basis $\{\mathbf{1}_x\}_{x\in X_\alpha}$, for every $\alpha\in\mathcal{J}$, because X_α is finite and discrete. As a topological space, $\mathcal{C}(X_\alpha,R)$ is homeomorphic to the product $R^{|X_\alpha|}$.

For all $\beta \geqslant \alpha$ in $\mathcal{J}$, we have a map

$$(\pi_\alpha^\beta)^*\colon \mathcal{C}(X_\alpha,R) \to \mathcal{C}(X_\beta,R)$$

by $f \mapsto f\circ\pi_\alpha^\beta$. Thus we get a direct system, with limit

$$\mathcal{L}_c(X,R) := \varinjlim_{\alpha\in\mathcal{J}} \mathcal{C}(X_\alpha,R)\,. \tag{5}$$

There is a continuous injection $\mathcal{L}_c(X,R) \hookrightarrow \mathcal{C}(X,R)$ induced by the maps

$$\pi_\alpha^*\colon \mathcal{C}(X_\alpha,R) \longrightarrow \mathcal{C}(X,R)\,,$$

$f \mapsto f\circ\pi_\alpha$.

Lemma 2.3. *The image of $\mathcal{L}_c(X,R)$ in $\mathcal{C}(X,R)$ consists exactly of the R-valued locally constant functions on X.*

Proof. By definition, the fibers $\pi_\alpha^{-1}(\pi_\alpha(x))$, as α varies in $\mathcal{J}$ and $x\in X$, form a basis of the topology. Hence, if f is locally constant, there is a cover of such fibers such that f is constant on each of them. By compactness, one can extract a finite subcover,

$$X = \bigcup_{i=1}^n \pi_{\alpha_i}^{-1}(\pi_{\alpha_i}(x_i)).$$

Take $\alpha\in\mathcal{J}$ such that $\alpha \geqslant \alpha_i$ for $i=1,\dots,n$. Then f factors through X_α. □

Corollary 2.4. *A subset of X is compact open if and only if it is α-saturated for some $\alpha\in\mathcal{J}$.*

Proof. If $U \subseteq X$ is compact open, then its characteristic function $\mathbf{1}_U$ is locally constant, hence one has $\mathbf{1}_U = f\circ\pi_\alpha$ for some α and $f\in\mathcal{C}(X_\alpha,R)$. This is possible only if $f = \mathbf{1}_S$ for some $S\subseteq X_\alpha$, with $U = \pi_\alpha^{-1}(S)$. Therefore $U = \pi_\alpha^{-1}(\pi_\alpha(U))$.

The converse implication is obvious from the definition of the profinite topology. □

Lemma 2.5. *The image of $\mathcal{L}_c(X,R)$ is dense in $\mathcal{C}(X,R)$.*

Proof. Let $f \in \mathcal{C}(X,R)$ and fix $U \subseteq R$ a neighbourhood of 0. By continuity, for any $x \in X$ we can find a neighbourhood A_x such that $f(y) - f(x) \in U$ for all $y \in A_x$. We can also assume $A_x = \pi_{\alpha_x}^{-1}(\pi_{\alpha_x}(x))$ for some $\alpha_x \in \mathcal{J}$. By compactness, we can extract a finite subcover $\{A_{x_1}, ..., A_{x_n}\}$ from the open cover $\{A_x\}$. Since $\mathcal{J}$ is directed, one can find an index $\alpha \geqslant \alpha_{x_i}$, $i = 1, ..., n$. For any $y \in X_\alpha$, choose $\tilde{y} \in \pi_\alpha^{-1}(y)$ and define a function f_U by

$$f_U(x) = \sum_{y \in X_\alpha} f(\tilde{y})\, \mathbf{1}_{\pi_\alpha^{-1}(y)} \,.$$

Then f_U is locally constant and $f(x) - f_U(x) \in U$ for every $x \in X$. □

By abuse of notation, in the following we will often identify $\mathcal{L}_c(X,R)$ and $\mathcal{C}(X_\alpha,R)$ with their images in $\mathcal{C}(X,R)$; note, however, that the topology on $\mathcal{L}_c(X,R)$ is not the one as a subspace of $\mathcal{C}(X,R)$. More precisely, the topology on $\mathcal{L}_c(X,R)$ is induced by (5): that is, $\mathcal{L}_c(X,R)$ is given the finest topology such that all the embeddings $\mathcal{C}(X_\alpha,R) \hookrightarrow \mathcal{L}_c(X,R)$ are continuous.

Finally, we observe that if R is complete then a standard argument shows that so is $\mathcal{C}(X,R)$. The same holds for $\mathcal{L}_c(X,R)$, with the direct limit topology, as we now prove.

Proposition 2.6. *If R is complete then so is $\mathcal{L}_c(X,R)$.*

Proof. We are going to show that, by definition of direct limit topology, a sequence $(f_n)_n$ is Cauchy in $\mathcal{L}_c(X,R)$ only if there is some α such that $f_n \in \mathcal{C}(X_\alpha,R)$ for every $n \gg 0$. Hence the completeness of $\mathcal{C}(X_\alpha,R)$ for every α implies that also $\mathcal{L}_c(X,R)$ is complete.

For $\alpha \in \mathcal{J}$, define a continuous function $m_\alpha \colon \mathcal{C}(X,R) \to \mathbb{R}$ by letting $\{U_{\alpha,i}\}$ be the partition of X into fibers of π_α and putting

$$m_\alpha(f) := \max_i \left\{ \sup_{x,y \in U_{\alpha,i}} |f(x) - f(y)|_R \right\} .$$

Then $\beta \geqslant \alpha$ implies $m_\alpha(f) \geqslant m_\beta(f)$ and one has $f \in \mathcal{C}(X_\alpha,R)$ if and only if $m_\alpha(f) = 0$.

Let $\mathcal{J}_0 = \{\alpha_n\}_{n \in \mathbb{N}}$ be a totally ordered countable cofinal subset of $\mathcal{J}$. Following [19, Theorem 6.5], it is enough to show that if $E \subseteq \mathcal{L}_c(X,R)$ is bounded then it is contained in $\mathcal{C}(X_\alpha,R)$ for some α. If not, then for every $\alpha_n \in \mathcal{J}_0$ there is $f_n \in E$ such that $m_{\alpha_n}(f_n) > 0$. Consider the set

$$W = \left\{ f \in \mathcal{L}_c(X,R) \mid m_{\alpha_n}(f) < \frac{m_{\alpha_n}(f_n)}{n} \;\forall\, n \right\} .$$

Given $\beta \in \mathcal{J}$, there is $n(\beta)$ such that $\alpha_n \geqslant \beta$ for every $n \geqslant n(\beta)$. This implies that

$$(\pi_\beta^*)^{-1}(W) = \left\{ f \in \mathcal{C}(X_\beta,R) \mid m_{\alpha_n}(f) < \frac{m_{\alpha_n}(f_n)}{n} \;\forall\, n < n(\beta) \right\}$$

is open. Therefore W is open and we have a contradiction, because $E \nsubseteq nW$ for all n. □

2.1.3. *Measures and distributions.* For a (pro)finite topological space, we define R-valued measures and distributions as the topological duals, respectively, of continuous and locally constant functions (so that they coincide when the space is finite). For X as in (4) we obtain the R-modules

$$\mathcal{M}(X,R) := \mathrm{Hom}_R(\mathcal{C}(X,R),R)$$

and

$$\mathcal{D}(X,R) := \mathrm{Hom}_R(\mathcal{L}_c(X,R),R) .$$

(In both cases, we only consider continuous homomorphisms.) From (5) one immediately gets

$$\mathcal{D}(X,R) = \varprojlim_{\alpha \in \mathcal{J}} \mathcal{D}(X_\alpha,R) \tag{6}$$

where the inverse system is built by the obvious maps

$$(\pi_\alpha^\beta)_* \colon \mathcal{D}(X_\beta,R) \to \mathcal{D}(X_\alpha,R) .$$

Any measure becomes a distribution when restricting it to locally constant functions.

Lemma 2.7. *The natural map $\mathcal{M}(X,R) \to \mathcal{D}(X,R)$ is injective.*

Proof. Assume μ_1 and μ_2 are different in $\mathcal{M}(X,R)$. Then there is $f \in \mathcal{C}(X,R)$ such that $\mu_1(f) \neq \mu_2(f)$. By Lemma 2.5 there is some sequence (f_n) of locally constant functions which converge uniformly to f. Since both μ_i's are continuous, we have

$$\mu_i(f) = \lim \mu_i(f_n)\,.$$

Thus there must be some locally constant f_k such that $\mu_1(f_k) \neq \mu_2(f_k)$. □

Lemma 2.8. *Let δ be an R-valued distribution on X. Assume that R is complete and $\delta(\mathbf{1}_U)$ is bounded as U varies among compact open subsets of X. Then δ is a measure on X.*

Proof. By Lemma 2.5, any $f \in \mathcal{C}(X,R)$ is the uniform limit of a sequence (f_n) of locally constant functions. The inequalities

$$|\delta(f_n) - \delta(f_m)|_R \leqslant \|f_n - f_m\|_\infty \cdot \sup_U |\delta(\mathbf{1}_U)|_R$$

show that the sequence $\delta(f_n)$ is Cauchy. Its limit will be $\delta(f)$. □

Note that $\mathcal{D}(X,R)$ is a module over the ring $\mathcal{L}_c(X,R)$, by

$$(f \cdot \delta)(g) := \delta(fg)\,. \tag{7}$$

2.1.4. *The topology of $\mathcal{D}(X,R)$.* Both $\mathcal{M}(X,R)$ and $\mathcal{D}(X,R)$ are endowed with the weak-$*$ topology (that is, the coarsest topology which makes all evaluation maps $ev_f\colon \mu \mapsto \mu(f)$ continuous, where f varies respectively in either $\mathcal{C}(X,R)$ or $\mathcal{L}_c(X,R)$). One easily checks that the R-module operations are continuous.

Lemma 2.9. *Let $\alpha \in \mathcal{J}$. The R-module $\mathcal{D}(X_\alpha,R)$ is free on the basis $\{\delta_x\}_{x\in X_\alpha}$, where δ_x is the Dirac functional at x. The weak-* topology on $\mathcal{D}(X_\alpha,R)$ is the same as the product topology as a free module.*

Proof. Recall Remark 2.2. The pairing $\bigoplus_{x\in X_\alpha} R\delta_x \times \mathcal{C}(X_\alpha,R) \to R$ given by

$$\left(\sum_x a_x\delta_x, \sum_x b_x\,\mathbf{1}_x\right) \mapsto \sum_x a_x b_x \tag{8}$$

is continuous in both variables because R is a topological ring. The first statement is an immediate consequence.

As for the second claim, it is clear from (8) that the product topology on $\mathcal{D}(X_\alpha,R)$ makes ev_f continuous for any $f \in \mathcal{C}(X_\alpha,R)$ and hence is finer than the weak-* topology. It is also coarser, because, by (8), the coordinate maps defining it are $\{ev_{\mathbf{1}_x}\}_{x\in X_\alpha}$, which must be continuous with respect to the weak-* topology. □

Lemma 2.9 yields an explicit description of the structure maps $(\pi_\alpha)_*\colon \mathcal{D}(X,R) \to \mathcal{D}(X_\alpha,R)$ from (6), namely

$$(\pi_\alpha)_*(\mu)\colon f = \sum_{x\in X_\alpha} a_x\,\mathbf{1}_x \mapsto \sum_{x\in X_\alpha} a_x\mu(\mathbf{1}_{\pi_\alpha^{-1}(x)}) = \sum_{x\in X_\alpha} \mu(\mathbf{1}_{\pi_\alpha^{-1}(x)})\delta_x(f)\,. \tag{9}$$

Proposition 2.10. *On $\mathcal{D}(X,R)$, the inverse limit topology from* (6) *is the same as the weak-* topology.*

Proof. Each of the two topologies is defined as the coarsest one which makes continuous the maps in a certain set, namely the evaluation maps $\{ev_f\}_{f\in\mathcal{L}_c(X,R)}$ in the weak-* case and the projections $\{(\pi_\alpha)_*\}_{\alpha\in\mathcal{J}}$ in the inverse limit case. So it is enough to show that if all the maps in one of the two sets are continuous then so are the ones in the other set and vice versa.

Take $f \in \mathcal{L}_c(X,R)$. Then, by (5), one has $f = \pi_\alpha^*(g)$ for some $g \in \mathcal{C}(X_\alpha,R)$ and the equality

$$\mu(f) = \mu(\pi_\alpha^*(g)) = (\pi_\alpha)_*(\mu)(g)$$

yields $ev_f = ev_g \circ (\pi_\alpha)_*$, with $ev_g\colon \mathcal{D}(X_\alpha,R) \to R$ the evaluation at g, which is continuous by Lemma 2.9. This proves that if all the projections are continuous then so are the evaluation maps.

On the other hand, (9) shows that the δ_x-coordinate of $(\pi_\alpha)_*$ is simply the evaluation at $\mathbf{1}_{\pi_\alpha^{-1}(x)}$. Therefore each $(\pi_\alpha)_*$ is continuous if all the evaluation maps are so. □

Corollary 2.11. *If R is complete, then so is $\mathcal{D}(X,R)$.*

Proof. An inverse limit of complete topological modules is complete. In this particular case, if $(\mu_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{D}(X,R)$, then each $((\pi_\alpha)_*(\mu_n))_{n\in\mathbb{N}}$ is also Cauchy, with limit μ_α. The equalities $(\pi_\alpha^\beta)_*(\mu_\beta)=\mu_\alpha$ hold (by continuity and uniqueness of the limit) for every $\beta\geqslant\alpha$, so there is $\mu\in\mathcal{D}(X,R)$ such that $(\pi_\alpha)_*(\mu)=\mu_\alpha$. Finally, $\mu=\lim_n\mu_n$ is true by construction. □

Remark 2.12. When X is a group, $\mathcal{D}(X,R)$ is an algebra with the convolution product. More precisely, $\mathcal{D}(X_\alpha,R)=\bigoplus_{x\in X_\alpha}R\delta_x$ is isomorphic to the group algebra $R[X_\alpha]$ and taking the limit one obtains

$$\mathcal{D}(X,R)\simeq R[[X]]:=\varprojlim_{\alpha\in\mathcal{J}}R[X_\alpha]\,.$$

In the case $R=\mathbb{Z}_p$ and X a p-adic Lie group, this additional structure plays an important role in Iwasawa theory.

2.1.5. *Products.* Assume we are given a collection of profinite sets $\{Y_\kappa\}_{\kappa\in\mathcal{K}}$, defined by $Y_\kappa=\varprojlim Y_{\kappa,\alpha_\kappa}$, where α_κ varies in a direct set $\mathcal{J}_\kappa$ and each Y_{κ,α_κ} is finite. We want to describe distributions on $\prod_\kappa Y_\kappa$.

Lemma 2.13. *Let $\{Y_\kappa\}_{\kappa\in\mathcal{K}}$ be a collection of profinite sets, as above, and put $X=\prod_{\kappa\in\mathcal{K}}Y_\kappa$. Consider the directed set $\mathcal{J}=\bigcup_K\mathcal{J}_\kappa$, where K varies among all finite subsets of $\mathcal{K}$ and the order is given by*

$$\beta=(\beta_\kappa)_{\kappa\in K'}\geqslant\alpha=(\alpha_\kappa)_{\kappa\in K}\ \text{ if } K\subseteq K' \text{ and } \beta_\kappa\geqslant\alpha_\kappa \text{ for all } \kappa\in K.$$

Then $X=\varprojlim_{\alpha\in\mathcal{J}}X_\alpha$, with $X_\alpha=\prod_{\kappa\in K}Y_{\kappa,\alpha_\kappa}$ for $\alpha=(\alpha_\kappa)_{\kappa\in K}\in\mathcal{J}$.

Proof. Obvious by abstract nonsense. □

Lemma 2.14. *Let $Y_1,\dots,Y_n$ be finite sets. There is an isomorphism of R-modules*

$$\bigotimes_{i=1}^n\mathcal{D}(Y_i,R)\longrightarrow\mathcal{D}(Y_1\times\cdots\times Y_n,R)$$

given by $\delta_{y_1}\otimes\cdots\otimes\delta_{y_n}\mapsto\delta_{(y_1,\dots,y_n)}$.

Proof. Straightforward from Lemma 2.9. □

Proposition 2.15. *Let $\{Y_\kappa\}_{\kappa\in\mathcal{K}}$ be a finite collection of profinite sets and put $X=\prod_{\kappa\in\mathcal{K}}Y_\kappa$. Then there is a natural morphism*

$$\bigotimes_{\kappa\in\mathcal{K}}\mathcal{D}(Y_\kappa,R)\longrightarrow\mathcal{D}(X,R)\tag{10}$$

with dense image.

Proof. We use the notation of Lemma 2.13. The isomorphism of Lemma 2.14 commutes with the structure maps induced by $\pi_{\kappa,\alpha_\kappa}\colon Y_\kappa\to Y_{\kappa,\alpha_\kappa}$ on the one side and $\pi_\alpha\colon X\to X_\alpha$ on the other. Taking the limit and composing with the natural morphism

$$\bigotimes_{\kappa\in\mathcal{K}}\mathcal{D}(Y_\kappa,R)=\bigotimes_{\kappa\in\mathcal{K}}\varprojlim\mathcal{D}(Y_{\kappa,\alpha_\kappa},R)\longrightarrow\varprojlim\bigotimes_{\kappa\in\mathcal{K}}\mathcal{D}(Y_{\kappa,\alpha_\kappa},R),$$

we obtain (10). The image is dense because the composition with $(\pi_\alpha)_*$ is surjective for every α, by Lemma 2.14. □

Remark 2.16. In order for (10) to be an isomorphism, one has to replace $\bigotimes\mathcal{D}(Y_\kappa,R)$ with $\widehat{\bigotimes}\mathcal{D}(Y_\kappa,R)$, the completed tensor product. For an explicit example of why this is needed, just consider the case $R=\mathbb{Z}_p$ and $Y_1=Y_2=\mathbb{Z}_p$. It is well-known that then one has $\mathcal{D}(Y_i,R)\simeq\mathbb{Z}_p[[t_i]]$ and $\mathcal{D}(Y_1\times Y_2,R)\simeq\mathbb{Z}_p[[t_1,t_2]]$. The natural map $\mathbb{Z}_p[[t_1]]\otimes\mathbb{Z}_p[[t_2]]\to\mathbb{Z}_p[[t_1,t_2]]$ is not surjective.

We shall denote the image via (10) of $\otimes_\kappa\mu_\kappa$ by the same symbol and call it the *product of the distributions* $\{\mu_\kappa\}_{\kappa\in\mathcal{K}}$.

Lemma 2.17. *Let X, Y_κ be as in Proposition 2.15. The product distribution $\otimes_\kappa\mu_\kappa$ is uniquely characterized by the property*

$$\otimes_\kappa\mu_\kappa\left(\mathbf{1}_{\prod U_\kappa}\right)=\prod_{\kappa\in\mathcal{K}}\mu_\kappa(\mathbf{1}_{U_\kappa})\,,\tag{11}$$

for any choice of compact open subsets $U_\kappa\subseteq Y_\kappa$.

Proof. By definition of the product topology, any compact open subset of X can be written as a finite union of sets of the form $\prod_\kappa U_\kappa$, with $U_\kappa \subseteq Y_\kappa$ compact open. Thus a distribution on X is determined by its values on the functions $\mathbf{1}_{\prod U_\kappa}$.

In order to check (11), there is no loss of generality in taking sets of the form $U_\kappa = \pi^{-1}_{\kappa,\alpha_\kappa}(x_\kappa)$, so to have $\prod U_\kappa = \pi_\alpha^{-1}(x)$, with $x = (x_\kappa)$. For all κ we have

$$\mu_\kappa(\mathbf{1}_{x_\kappa}) = \big((\pi_\alpha)_*(\mu_\kappa)\big)(\mathbf{1}_{x_\kappa}) = \sum_{y_\kappa \in Y_{\kappa,\alpha_\kappa}} c_{y_\kappa}\delta_{y_\kappa}(\mathbf{1}_{x_\kappa}) = c_{x_\kappa}$$

$$\otimes_\kappa\mu_\kappa\big(\mathbf{1}_{\pi_\alpha^{-1}(x)}\big) \stackrel{\dagger}{=} \big((\pi_\alpha)_*(\otimes_\kappa\mu_\kappa)\big)(\mathbf{1}_x) = \left(\bigotimes\left(\sum_{y_\kappa \in Y_{\kappa,\alpha_\kappa}} c_{y_\kappa}\delta_{y_\kappa}\right)\right)(\mathbf{1}_{x_\kappa}) \stackrel{\ddagger}{=} \prod c_{x_\kappa}$$

where † applies (9) and ‡ follows from Lemma 2.14. □

Equality (11) shows the problem in extending Proposition 2.15 to the case when the set of indexes $\mathcal{K}$ is infinite. Indeed, taking $\mu_\kappa \in \mathcal{D}(Y_\kappa, R)$ such that $\mu_\kappa(\mathbf{1}_{Y_\kappa}) = c$ for every κ, where $c \in R$ is such that the sequence c^n has no limit, then we see no way of giving to $(\otimes_\kappa\mu_\kappa)(\mathbf{1}_X)$ a meaning compatible with (11).

We say that an infinite product $\prod_{\kappa\in\mathcal{K}} c_\kappa$ *converges in* R if the equality $\lim_K \prod_{\kappa\in K} c_\kappa = r$ (where the limit is taken for K varying among finite subsets of $\mathcal{K}$) is satisfied for some $r \in R$.

Theorem 2.18. *Let $X = \prod_{\kappa\in\mathcal{K}} Y_\kappa$, where each Y_κ is profinite. For each $\kappa \in \mathcal{K}$ let $\mu_\kappa \in \mathcal{D}(Y_\kappa, R)$ be such that $\prod_{\kappa\in\mathcal{K}} \mu_\kappa(\mathbf{1}_{Y_\kappa})$ converges. Then the product distribution $\otimes_\kappa\mu_\kappa$ exists.*

Proof. Let U be a compact open subset of X. Without loss of generality we can assume

$$U = \prod_{\kappa\in K} U_\kappa \times \prod_{\kappa\in\mathcal{K}-K} Y_\kappa \tag{12}$$

where $K \subseteq \mathcal{K}$ is finite and each $U_\kappa \subseteq Y_\kappa$ compact open. Thus we can follow (11) and define

$$\otimes_\kappa\mu_\kappa(\mathbf{1}_U) := \prod_{\kappa\in K} \mu_\kappa(\mathbf{1}_{U_\kappa}) \times \prod_{\kappa\in\mathcal{K}-K} \mu_\kappa(\mathbf{1}_{Y_\kappa}). \tag{13}$$

The hypothesis implies that the product on the right-hand side of (13) converges to a limit which is independent of the choice of the decomposition of U. □

Remark 2.19. By the above definition of convergence, we allowed $\lim_K \prod_{\kappa\in K} \mu_\kappa(Y_\kappa) = 0$. When this happens, $\otimes_\kappa\mu_\kappa$ need not be trivial: for example, fix a finite set K_0 and choose the distributions μ_κ so that $\mu_\kappa(\mathbf{1}_{Y_\kappa})$ is 0 if $\kappa \in K_0$ and 1 otherwise. Distributions with total mass 0 play an important role in certain parts of non-archimedean analysis, so it seems more convenient to keep this possibility open also for our definition of product distribution.

Let X be a profinite set and $Z \subseteq X$ a closed subset. A distribution $\mu \in \mathcal{D}(X, R)$ is Z*-normalized* if $\mu(\mathbf{1}_Z) = 1$. Denote

$$\mathcal{D}_1(X, R) := \{\mu \in \mathcal{D}(X, R) \mid \mu(\mathbf{1}_X) = 1\}$$

Corollary 2.20. *The map $\mu \mapsto \big(\pi_{\kappa*}(\mu)\big)$ induces a bijection from $\mathcal{D}_1(X, R)$ to $\prod_{\kappa\in\mathcal{K}} \mathcal{D}_1(Y_\kappa, R)$.*

Proof. □

2.2. **Procounting distributions.** From now on we assume that R is a $\mathbb{Q}$-algebra, so to have

$$\frac{1}{|X_\alpha|} \in R \ \forall\, \alpha \in \mathcal{J}.$$

2.2.1. *A categorical limit.* We define:

$$\mu_\alpha := \frac{1}{|X_\alpha|} \sum_{x\in X_\alpha} \delta_x \in \mathcal{D}(X_\alpha, R). \tag{14}$$

This is a distribution on X_α.

Definition 2.21. The *procounting distribution* on X is

$$\mu_X := \varprojlim_{\alpha\in\mathcal{J}} \mu_\alpha \in \mathcal{D}(X,R)$$

(when this limit exists).

Proposition 2.22. *The inverse limit μ_X exists as a distribution in $\mathcal{D}(X,R)$ if and only if*

$$|(\pi_\alpha^\beta)^{-1}(x)| = \frac{|X_\beta|}{|X_\alpha|} \quad \text{for all } x\in X_\alpha\,, \beta\geqslant\alpha \text{ in } \mathcal{J}. \tag{15}$$

Proof. By definition of inverse limit, μ_X exists if and only if the equality

$$\mu_\alpha = (\pi_\alpha^\beta)_*(\mu_\beta) \tag{16}$$

is satisfied for every $\beta\geqslant\alpha$ in $\mathcal{J}$. By (14), condition (16) can be rewritten as

$$\frac{1}{|X_\alpha|}\sum_{x\in X_\alpha}\delta_x = \mu_\alpha = (\pi_\alpha^\beta)_*(\mu_\beta) = \frac{1}{|X_\beta|}\sum_{y\in X_\beta}(\pi_\alpha^\beta)_*(\delta_y) = \frac{1}{|X_\beta|}\sum_{y\in X_\beta}\delta_{\pi_\alpha^\beta(y)} =$$

$$= \frac{1}{|X_\beta|}\sum_{x\in X_\alpha}\sum_{y\in(\pi_\alpha^\beta)^{-1}(x)}\delta_x = \frac{1}{|X_\beta|}\sum_{x\in X_\alpha}|(\pi_\alpha^\beta)^{-1}(x)|\delta_x$$

The equivalence between (15) and (16) now follows by Lemma 2.9. □

Remarks 2.23.
1. Condition (15) is obviously satisfied when X is a profinite group. In this case the inverse limit μ_X is precisely the Haar measure on X.
2. Assume μ_X exists and let $U\subseteq X$ be compact open. Then $U=\pi_\alpha^{-1}(\pi_\alpha(U))$ for some α, by Corollary 2.4, and thus $\mathbf{1}_U = \mathbf{1}_{\pi_\alpha(U)}\circ\pi_\alpha$. Definition 2.21 yields

$$\mu_\alpha = (\pi_\alpha)_*(\mu_X) = \mu_X\circ(\pi_\alpha)^*\,,$$

which, together with (14), implies

$$\mu_X(\mathbf{1}_U) = \mu_X((\pi_\alpha)^*\,\mathbf{1}_{\pi_\alpha(U)}) = \mu_\alpha(\mathbf{1}_{\pi_\alpha(U)}) = \frac{1}{|X_\alpha|}\sum_{x\in X_\alpha}\delta_x(\mathbf{1}_{\pi_\alpha(U)}) = \frac{|\pi_\alpha(U)|}{|X_\alpha|}\,. \tag{17}$$

3. In general, we cannot expect μ_X to be a measure: indeed, it is well-known that $\mu_{\mathbb{Z}_p}$ is a distribution and not a measure when $R=\mathbb{Q}_p$ (see below for a proof). However μ_X is always a measure in the cases of most interest for this paper, with R either $\mathbb{R}$ or $\mathbb{C}$, as the following proposition proves.

Proposition 2.24. *Assume μ_X esists. It is a measure only if the set $\{\frac{1}{|X_\alpha|}\}_{\alpha\in\mathcal{J}}$ is bounded in R. This condition is also sufficient if R is complete.*

Proof. The last statement is a straightforward consequence of formula (17) and Lemma 2.8 (using the fact that $\mathbb{N}$ is bounded in R if $|\cdot|_R$ is non-archimedean and $\big||\pi_\alpha(U)|\big|_R \leqslant \big||X_\alpha|\big|_R$ otherwise). As for necessity, assume that $\{|X_\alpha|^{-1}\}_{\alpha\in\mathcal{J}}$ is unbounded. Then there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}\subseteq\mathcal{J}$ such that $|X_{\alpha_n}|=c_n$ with

$$|c_n|_R < \frac{1}{n}\,. \tag{18}$$

Fix $z\in X$ and consider

$$f_n := c_n\,\mathbf{1}_{\pi_{\alpha_n}^{-1}(\pi_{\alpha_n}(z))} = g_n\circ\pi_{\alpha_n}$$

with $g_n = c_n\,\mathbf{1}_{\pi_{\alpha_n}(z)}$. The sequence $(f_n)_{n\in\mathbb{N}}$ converges to 0 in $\mathcal{C}(X,R)$, by (18). However, reasoning as in (17) shows

$$\mu_X(f_n) = \mu_X((\pi_{\alpha_n})^*g_n) = \mu_{\alpha_n}(g_n) = \frac{1}{|X_{\alpha_n}|}\sum_{x\in X_{\alpha_n}}\delta_x(c_n\,\mathbf{1}_{\pi_{\alpha_n}(z)}) = \frac{1}{c_n}\cdot c_n = 1$$

for every n. Thus μ_X cannot extend to a continuous functional on $\mathcal{C}(X,R)$. □

2.2.2. *The Hecke submodule.* Assume that μ_X exists (for example because X is a group). Then we can use (7) to define a map

$$\mathcal{H}\colon \mathcal{L}_c(X,R) \longrightarrow \mathcal{D}(X,R)$$

by $f \mapsto f \cdot \mu_X$. We define the *Hecke submodule* to be the image of $\mathcal{H}$. The equality

$$(\pi_\alpha)_*(\mathbf{1}_{\pi_\alpha^{-1}(x)}) = \delta_x$$

implies that the Hecke submodule surjects onto all $\mathcal{D}(X_\alpha,R)$ and hence is dense in $\mathcal{D}(X,R)$.

Remark 2.25. When X is a group, the R-module $\mathcal{D}(X,R)$ becomes an algebra with the convolution product. Readers might be familiar with the case $X = \mathrm{GL}_n(\mathbf{A}_K^{(\infty)})$, where K is a global field and $\mathbf{A}_K^{(\infty)}$ the non-archimedean part of its adele ring: in this case, the Hecke submodule is the "usual" Hecke algebra, as appearing in the theory of automorphic forms.

The distribution functor is covariant: if $\phi\colon X \to Y$ is a continuous map of (pro)finite sets, it induces $\phi_*\colon \mathcal{D}(X,R) \to \mathcal{D}(Y,R)$ by $\phi_*(\mu)(f) = \mu(f \circ \phi)$ (because $f \circ \phi$ is locally constant if so is f). However, restriction to the Hecke submodule yields a contravariant functor: in particular, this allows us to lift distributions from X_α to X. The idea is summarized by the commutative diagram

$$\begin{tikzcd}
\mathcal{L}_c(X,R) \arrow[r, "\mathcal{H}"] & \mathcal{D}(X,R) \arrow[d, "(\pi_\alpha)_*"] \\
\mathcal{C}(X_\alpha,R) \arrow[u, "(\pi_\alpha)^*"] \arrow[r, "\mathcal{H}_\alpha"'] & \mathcal{D}(X_\alpha,R)
\end{tikzcd}$$

(where $\mathcal{H}_\alpha$ is defined in the obvious way). The equality $(\pi_\alpha)_* \circ \mathcal{H} \circ (\pi_\alpha)^* = \mathcal{H}_\alpha$ can be checked by computing

$$(\pi_\alpha)_*\big((\pi_\alpha)^*(\mathbf{1}_x)\cdot\mu_X\big) = (\pi_\alpha)_*(\mathbf{1}_{\pi_\alpha^{-1}(x)}\cdot\mu_X) = \mathbf{1}_{\pi_\alpha(\pi_\alpha^{-1}(x))}\cdot(\pi_\alpha)_*(\mu_X) = \mathbf{1}_x\cdot\mu_\alpha = \frac{1}{|X_\alpha|}\delta_x\,.$$

2.2.3. *A topological limit.* As above, assume that μ_X exists. For $S \subseteq X$, let

$$\mu_{S,\alpha} := \frac{1}{|\pi_\alpha(S)|}\sum_{x\in\pi_\alpha(S)}\delta_x \in \mathcal{D}(X_\alpha,R)\,.$$

If S is closed we have $S = \varprojlim \pi_\alpha(S)$. However, there is no reason why the counting distributions $\mu_{S,\alpha}$ should satisfy condition (15) and form an inverse system. Define

$$\tilde{\mu}_{S,\alpha} := \frac{|X_\alpha|}{|\pi_\alpha(S)|}\sum_{x\in\pi_\alpha(S)}\mathbf{1}_{\pi_\alpha^{-1}(x)}\cdot\mu_X = \frac{|X_\alpha|}{|\pi_\alpha(S)|}\,\mathbf{1}_{\pi_\alpha^{-1}(\pi_\alpha(S))}\cdot\mu_X \in \mathcal{D}(X,R)\,, \tag{19}$$

so that

$$\begin{aligned}
(\pi_\alpha)_*(\tilde{\mu}_{S,\alpha}) &= \frac{|X_\alpha|}{|\pi_\alpha(S)|}(\pi_\alpha)_*(\mathbf{1}_{\pi_\alpha^{-1}(\pi_\alpha(S))}\cdot\mu_X) = \frac{|X_\alpha|}{|\pi_\alpha(S)|}\;\mathbf{1}_{\pi_\alpha(S)}\cdot\mu_\alpha = \\
&= \frac{|X_\alpha|}{|\pi_\alpha(S)|}\,\frac{1}{|X_\alpha|}\sum_{x\in\pi_\alpha(S)}\delta_x = \mu_{S,\alpha}
\end{aligned}$$

In particular, if U is α-saturated one has

$$\tilde{\mu}_{S,\alpha}(\mathbf{1}_U) = \mu_{S,\alpha}(\mathbf{1}_{\pi_\alpha(U)}) = \frac{|\pi_\alpha(U)\cap\pi_\alpha(S)|}{|\pi_\alpha(S)|} \tag{20}$$

by the same reasoning as used in (17).

Definition 2.26. Let S be a subset of X. The *procounting distribution* μ_S attached to S is the limit (if it exists)

$$\mu_S = \lim_{\alpha\in\mathcal{J}}\tilde{\mu}_{S,\alpha} \tag{21}$$

of the net $\tilde{\mu}_{S,\alpha}$ in $\mathcal{D}(X,R)$.

The limit in (21) is taken with respect to the topology of $\mathcal{D}(X,R)$, as discussed in §2.1.4. Thus $\mu_S = \lim_\alpha \tilde{\mu}_{S,\alpha}$ means that for every neighborhood $\mathcal{U}$ of μ_S one can find an index α_0 such that $\tilde{\mu}_{S,\alpha} \in \mathcal{U}$ if $\alpha > \alpha_0$. In practice, it is enough to check the convergence of the values $\tilde{\mu}_{S,\alpha}(\mathbf{1}_U)$ for every compact open $U \subseteq X$.

Remark 2.27. Definitions 2.21 and 2.26 are compatible: if μ_X exists in the sense of Definition 2.21 then it is also the limit of the net $\tilde{\mu}_{X,\alpha}$. Indeed, (19) yields $\tilde{\mu}_{X,\alpha} = \mu_X$ for every α.

Lemma 2.28. *Let $\overline{S}$ be the closure of $S \subseteq X$. Then μ_S exists if and only if so does $\mu_{\overline{S}}$. Moreover, these two distributions are equal.*

Proof. Lemma 2.1 immediately yields that the equality $\pi_\alpha(S) = \pi_\alpha(\overline{S})$ holds for every $\alpha \in \mathcal{J}$. By (19), this implies $\tilde{\mu}_{S,\alpha} = \tilde{\mu}_{\overline{S},\alpha}$. □

By Lemma 2.28, in the following we shall mostly consider closed S. If S is also open, the situation is particularly nice, as the next result shows.

Lemma 2.29. *Assume $S = \pi_\gamma^{-1}(\pi_\gamma(S))$ for some $\gamma \in \mathcal{J}$. Then $\tilde{\mu}_{S,\alpha}$ converges to $\mu_S = c_S\,\mathbf{1}_S \cdot \mu_X$ where*

$$c_S = \frac{|X_\gamma|}{|\pi_\gamma(S)|} = \frac{1}{\mu_X(\mathbf{1}_S)}\,.$$

Proof. Since μ_X exists, all fibers of the transition maps π_α^β have the same cardinality, by Proposition 2.22. The equality $\pi_\alpha(S) = (\pi_\gamma^\alpha)^{-1}(\pi_\gamma(S))$ holds for $\alpha \geqslant \gamma$, by the assumption on S. Thus (15) yields

$$\frac{|X_\alpha|}{|X_\gamma|} = \frac{|\pi_\alpha(S)|}{|\pi_\gamma(S)|}\,.$$

and hence $c_S = |X_\alpha|/|\pi_\alpha(S)|$ for $\alpha \geqslant \gamma$. Therefore (19) becomes $\tilde{\mu}_{S,\alpha} = c_S\,\mathbf{1}_S \cdot \mu_X$ for any such α. □

We conclude with an example where μ_S does not exist.

Example 2.30. For p be an odd prime, take $X = \mathbb{Z}_p$ (with defining maps $\pi_n \colon \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$) and $S = S_0 \cup S_1$, with

$$S_0 := \left\{ \sum_{k=0}^{\infty} a_k p^k \mid a_k = 0 \text{ if } k \text{ is even }, a_k \in \{1, \dots, p-1\} \text{ if } k \text{ is odd} \right\}$$

and

$$S_1 := \left\{ \sum_{k=0}^{\infty} a_k p^k \mid a_k = 0 \text{ if } k \text{ is odd }, a_k \in \{1, \dots, p-1\} \text{ if } k \text{ is even} \right\}.$$

Thus one has $\pi_1(S_0) = \{[0]_p\}$, $\pi_1(S_1) = (\mathbb{Z}/p\mathbb{Z})^*$ and, for $x \in \pi_n(S_0)$,

$$|(\pi_n^{n+1})^{-1}(x)| = \begin{cases} p-1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

while the reverse holds for $x \in \pi_n(S_1)$. A simple induction then shows

$$|\pi_n(S_0)| = (p-1)^{\lfloor n/2 \rfloor} \quad \text{and} \quad |\pi_n(S_1)| = (p-1)^{\lfloor (n+1)/2 \rfloor}\,.$$

Since $\pi_n(S) \cap \pi_n(p\mathbb{Z}_p) = \pi_n(S_0)$, formula (20) yields

$$\tilde{\mu}_{S,n}(\mathbf{1}_{p\mathbb{Z}_p}) = \frac{|\pi_n(S_0)|}{|\pi_n(S)|} = \begin{cases} \dfrac{1}{p} & \text{if } n \text{ is odd,} \\ \dfrac{1}{2} & \text{if } n \text{ is even.} \end{cases}$$

This proves that the distributions $\tilde{\mu}_{S,n}$ do not converge (independently of the choice of R).

2.2.4. *The ambient space.* In §2.2.3, the distribution μ_S was defined assuming the existence of μ_X. Now we will show that this definition is in fact independent of the ambient space.

If $S \subseteq X$ is closed then one has $S = \varprojlim_{\alpha \in \mathcal{J}} \pi_\alpha(S)$. The inclusions $\iota_\alpha \colon \pi_\alpha(S) \hookrightarrow X_\alpha$ induce

$$\mathcal{D}(\pi_\alpha(S), R) \longhookrightarrow \mathcal{D}(X_\alpha, R) \quad \forall\, \alpha \in \mathcal{J}$$

and thus, taking the limit, $\iota_* \colon \mathcal{D}(S,R) \longhookrightarrow \mathcal{D}(X,R)$.

Lemma 2.31. *In the setting above, the map ι_* is a closed embedding, with image*

$$\iota_*\big(\mathcal{D}(S,R)\big) = \big\{\mu \in \mathcal{D}(X,R) \mid \mu(\mathbf{1}_U) = 0 \text{ if } U \cap S = \emptyset\big\}$$

where U varies among all the compact open subsets of X.

Proof. Lemma 2.9 immediately yields that the image of $(\iota_\alpha)_*$ is the space

$$\big\{\mu \in \mathcal{D}(X_\alpha, R) \mid \mu(\mathbf{1}_x) = 0 \text{ if } x \notin \pi_\alpha(S)\big\}$$

with basis $\{\delta_x \mid x \in \pi_\alpha(S)\}$. Hence the maps $(\iota_\alpha)_*$ are all closed embeddings. This implies the same for ι_*.

Let $\mu \in \iota_*\big(\mathcal{D}(S,R)\big)$ and fix a compact open $U \subseteq X$. Since S is closed, Lemma 2.1 shows that $U \cap S = \emptyset$ holds only if there is some $\alpha \in \mathcal{J}$ such that $\pi_\alpha(S) \cap \pi_\alpha(U) = \emptyset$. Replacing, if needed, α with a bigger index, we can assume that U is α-saturated: but then

$$\mu(\mathbf{1}_U) = \mu(\mathbf{1}_{\pi_\alpha^{-1}(\pi_\alpha(U))}) = (\pi_\alpha)_*(\mu)(\mathbf{1}_{\pi_\alpha(U)}) = 0,$$

because $(\pi_\alpha)_*(\mu)$ is in $\iota_*\big(\mathcal{D}(\pi_\alpha(S), R)\big)$.

Vice versa, if $\mu(\mathbf{1}_U) = 0$ for every compact open U, it follows

$$(\pi_\alpha)_*(\mu)(\mathbf{1}_x) = \mu(\mathbf{1}_{\pi_\alpha^{-1}(x)}) = 0,$$

showing that $(\pi_\alpha)_*(\mu)$ is in the image of $(\iota_\alpha)_*$ for all α and hence $\mu \in \mathcal{D}(S,R)$. □

In the following, we shall identify $\mathcal{D}(S,R)$ with its image via ι_*.

Corollary 2.32. *If μ_S exists, then it is the unique element in $\mathcal{D}(S,R)$ such that*

$$\mu_S(\mathbf{1}_U) = \lim_\alpha \frac{|\pi_\alpha(U)|}{|\pi_\alpha(S)|} \quad \text{for any compact open } U \subseteq S. \tag{22}$$

Proof. By definition, we have $\mu_S(\mathbf{1}_U) = \lim_\alpha \tilde{\mu}_{S,\alpha}(\mathbf{1}_U)$. Now just apply (20) and Lemma 2.31. □

Remark 2.33. Corollary 2.32 makes it clear that μ_S is independent of the ambient space X. The intrinsic characterization of μ_S provided by (22) could be used as definition of the procounting distribution attached to S, in alternative to (21). We have chosen to start with the former because in the situations of interest to us there is always a natural ambient space X such that μ_X exists in the sense of Definition 2.21.

Finally, we note that the assumption about the existence of μ_X can be made without any loss of generality.

Proposition 2.34. *For any profinite space X as in* (4) *there exists a closed embedding in a profinite space Y such that μ_Y exists (in the sense of Definition 2.21).*

Sketch of proof. Let $\mathcal{J}_0 = \{\alpha_n\}_{n \in \mathbb{N}} \subseteq \mathcal{J}$ be a countable cofinal chain. We shall use the shortenings X_n for X_{α_n} and so on.

The idea is to construct Y by embedding each X_n into a set Y_n so that for every n the fibers of $\pi_n^{n+1} \colon Y_{n+1} \to Y_n$ have all the same cardinality. This can be achieved starting with $Y_0 = X_0$ and then defining recursively Y_n by adding enough points to X_n. The existence of μ_Y then follows from Proposition 2.22. □

2.2.5. *The procounting distribution of a product.* Let $X = \prod_{\kappa\in\mathcal{K}} Y_\kappa$ be as in §2.1.5 and consider a subset of the form $S = \prod_\kappa T_\kappa$. By Lemma 2.13, X is the limit of X_α with $\alpha = (\alpha_\kappa)_{\kappa\in K} \in \bigcup \mathcal{J}_K$. One has

$$\pi_\alpha(S) = \prod_{\kappa\in K} \pi_{\alpha_\kappa}(T_\kappa)$$

and hence the isomorphism of Lemma 2.14 yields

$$\bigotimes_{\kappa\in K} \mu_{T_\kappa,\alpha_\kappa} = \bigotimes_{\kappa\in K} \sum_{y\in\pi_{\alpha_\kappa}(T_\kappa)} \frac{\delta_y}{|\pi_{\alpha_\kappa}(T_\kappa)|} \mapsto \frac{1}{|\pi_\alpha(S)|} \sum_{x\in\pi_\alpha(S)} \delta_x = \mu_{S,\alpha} . \tag{23}$$

Proposition 2.35. *Assume $X = \prod_{\kappa\in\mathcal{K}} Y_\kappa$, as in Proposition* **??***, and consider a subset of the form $S = \prod_\kappa T_\kappa$. If the procounting distribution μ_{T_κ} exists for every index κ, then μ_S also exists and is the product of the μ_{T_κ}'s.*

Proof. Since, by definition, $\mu_{T_\kappa}(\mathbf{1}_{T_\kappa}) = 1$ hold for every κ, Theorem 2.18 ensures the existence of $\otimes_\kappa \mu_{T_\kappa}$ and we just have to check that the latter satisfies Definition 2.26.

Let $U \subseteq X$ be compact open. By Corollary 2.4 and Lemma 2.13, U is α-saturated for some $\alpha = (\alpha_\kappa)_{\kappa\in K_0}$, where K_0 is a finite subset of $\mathcal{K}$. If $\beta \geqslant \alpha$, one has $\beta = (\beta_\kappa)_{\kappa\in K}$ with $K_0 \subseteq K$ and $\beta_\kappa \geqslant \alpha_\kappa$ for all $\kappa \in K_0$. Thus for such a β, we obtain

$$\pi_\beta(U) = \prod_{\kappa\in K} \pi_{\beta_\kappa}(U) = \prod_{\kappa\in K_0} \pi_{\beta_\kappa}(U) \times \prod_{\kappa\in K-K_0} Y_{\kappa,\beta_\kappa} ,$$

which, together with (23), implies

$$\mu_{S,\beta}(\mathbf{1}_{\pi_\beta(U)}) = \prod_{\kappa\in K_0} \frac{|\pi_{\beta_k}(U) \cap \pi_{\beta_\kappa}(T_\kappa)|}{|\pi_{\beta_\kappa}(T_\kappa)|} \cdot \prod_{\kappa\in K-K_0} 1 = \prod_{\kappa\in K_0} \mu_{T_\kappa,\beta_\kappa}(\mathbf{1}_{\pi_{\beta_k}(U)}) \tag{24}$$

As a finite product of convergent terms, the right-hand side of (24) has a limit as β grows. This proves that μ_S exists and is given by the formula

$$\mu_S(\mathbf{1}_U) = \prod_{\kappa\in\mathcal{K}} \mu_{T_\kappa}(\mathbf{1}_{\pi_\kappa(U)}) \tag{25}$$

where $\pi_\kappa \colon X \to Y_\kappa$ is the canonical projection. (Note that our proof shows that $\pi_\kappa(U) = Y_\kappa$ for $\kappa \notin K_0$. Hence $\mu_{T_\kappa}(\mathbf{1}_{\pi_\kappa(U)}) = 1$ for almost every κ and the right-hand side of (25) is a finite product.) □

2.3. **Procounting measures.** From now on, we take $\mathbb{R}$ (or $\mathbb{C}$), with the usual absolute value, as our ring of coefficients R. Proposition 2.24 implies that if μ_X exists then it can be extended to a positive functional on $\mathcal{C}(X,\mathbb{R})$ and hence, by the Riesz representation theorem [18, 2.14], it defines a regular Borel measure (i.e., a Radon measure), which we shall still denote as μ_X. In particular, if $C \subseteq X$ is closed we have

$$\mu_X(C) = \inf_{C\subseteq U} \mu_X(\mathbf{1}_U) = \inf_{\alpha\in\mathcal{J}_0} \mu_X(\mathbf{1}_{\pi_\alpha^{-1}(\pi_\alpha(C))})$$

by Lemma 2.1 (where U varies among compact open subsets).

Lemma 2.36. *If S is open, then μ_S exists and it satisfies $\mu_S = \sup_U \mu_U$, where U varies among all the compact open subsets of X contained in S.*

Proof. For any $\alpha \in \mathcal{J}$, let

$$A_\alpha := \{x \in X_\alpha \mid \pi_\alpha^{-1}(x) \subseteq S\} \tag{26}$$

and $U_\alpha := \pi_\alpha^{-1}(A_\alpha)$. Each U_α is compact open and, as α grows in $\mathcal{J}$, they form an increasing cover of S (since the latter is open).

By definition of the topology on X, the hypothesis that yields $S = \cup_\alpha \pi_\alpha^{-1}(A_\alpha)$. Therefore we obtain, for any compact open $V \subseteq X$,

$$\lim_{\alpha\in\mathcal{J}} \mu_{S,\alpha}(\mathbf{1}_{\pi_\alpha(V)}) = \lim_{\alpha\in\mathcal{J}} \frac{|\pi_\alpha(V) \cap A_\alpha| + |\pi_\alpha(V) \cap B_\alpha|}{|A_\alpha| + |B_\alpha|} = \lim_{\alpha\in\mathcal{I}} \frac{|\pi_\alpha(V) \cap A_\alpha|}{|A_\alpha|} . \tag{27}$$

Put. One concludes observing that, for α big enough (so that V is α-saturated), the term in the limit on the right-hand side of (27) is $\mu_{U_\alpha}(\mathbf{1}_V)$. □

2.3.1. *Close pairs of subsets.*

Definition 2.37. We say that two subsets S, T of X *form a close pair* if

$$\lim_{\alpha\in\mathcal{J}} \frac{|\pi_\alpha(S)\Delta\pi_\alpha(T)|}{|\pi_\alpha(S\cup T)|} = 0 \tag{28}$$

where Δ denotes the symmetric difference.

Example 2.38. Lemma 2.1 implies that the image under π_α of a subset and of its closure are the same, for every $\alpha \in \mathcal{J}$. Therefore any $S \subseteq X$ forms a close pair with its own closure.

Remark 2.39. Instead of checking (28), it might be more convenient to consider the equivalent condition

$$\lim_{\alpha\in\mathcal{J}} \frac{|\pi_\alpha(S)\Delta\pi_\alpha(T)|}{|\pi_\alpha(S)\cap\pi_\alpha(T)|} = 0. \tag{29}$$

Lemma 2.28 has the following generalization.

Proposition 2.40. *If S and T form a close pair, then either they have the same procounting measure or both their procounting measures do not exist.*

Proof. For $\alpha \in \mathcal{J}$, put $B_\alpha = \pi_\alpha(S) \cap \pi_\alpha(T)$, $A_\alpha = \pi_\alpha(S) - B_\alpha$ and $C_\alpha = \pi_\alpha(T) - B_\alpha$. Thus for a compact open $V \subseteq X$ we have

$$\mu_{S,\alpha}(\mathbf{1}_{\pi_\alpha(V)}) = \frac{|\pi_\alpha(V)\cap A_\alpha|}{|A_\alpha|+|B_\alpha|} + \frac{|\pi_\alpha(V)\cap B_\alpha|}{|A_\alpha|+|B_\alpha|}$$

and

$$\mu_{T,\alpha}(\mathbf{1}_{\pi_\alpha(V)}) = \frac{|\pi_\alpha(V)\cap C_\alpha|}{|C_\alpha|+|B_\alpha|} + \frac{|\pi_\alpha(V)\cap B_\alpha|}{|C_\alpha|+|B_\alpha|}$$

Moreover, the closeness hypothesis can be rewritten as

$$\lim_{\alpha\in\mathcal{J}} \frac{|A_\alpha|+|C_\alpha|}{|A_\alpha|+|B_\alpha|+|C_\alpha|} = 0\,,$$

which implies

$$\lim_{\alpha\in\mathcal{J}} \frac{|A_\alpha|}{|B_\alpha|} = 0 = \lim_{\alpha\in\mathcal{J}} \frac{|C_\alpha|}{|B_\alpha|}$$

and therefore

$$\lim_{\alpha\in\mathcal{J}} \mu_{S,\alpha}(\mathbf{1}_{\pi_\alpha(V)}) = \lim_{\alpha\in\mathcal{J}} \frac{|\pi_\alpha(V)\cap B_\alpha|}{|B_\alpha|} = \lim_{\alpha\in\mathcal{J}} \mu_{T,\alpha}(\mathbf{1}_{\pi_\alpha(V)})$$

if the limits exist. □

Let $\partial_X S$ denote the boundary of S as a subset of X.

Proposition 2.41. *If $\mu_X(\partial S) = 0 \neq \mu_X(S)$, then S and its interior form a close pair.*

Proof. By Lemma 2.28, we can assume that S is closed. For any $\alpha \in \mathcal{J}$, let

$$A_\alpha := \{x \in X_\alpha \mid \pi_\alpha^{-1}(x) \subseteq S\}$$

and $B_\alpha := \pi_\alpha(S) - A_\alpha$. Then $A := \cup_\alpha \pi_\alpha^{-1}(A_\alpha)$ is the interior of S, while $\cap_\alpha \pi_\alpha^{-1}(B_\alpha) = \partial S$. The hypotheses then imply

$$0 < \mu_X(S) = \lim_{\alpha\in\mathcal{J}} \frac{|A_\alpha|+|B_\alpha|}{|X_\alpha|}$$

and

$$0 = \mu_X(\partial S) = \lim_{\alpha\in\mathcal{J}} \frac{|B_\alpha|}{|X_\alpha|}\,,$$

(because $\beta \geqslant \alpha$ implies $B_\beta \subseteq B_\alpha$) which, together, yield

$$\lim_{\alpha\in\mathcal{J}} \frac{|B_\alpha|}{|A_\alpha|} = 0.$$

□

Remark 2.42. Since the boundary has measure zero, the condition $\mu_X(S) \neq 0$ becomes equivalent to stating that S has nonempty interior as a subset of X. If this were to fail, then $\sup_U \mu_U$ would be trivially zero. However there are many interesting cases of sets whose closure has empty interior and to which one can attach a nontrivial measure: one case we are going to discuss in detail is the set of primes $\mathcal{P}$ in $X = \widehat{\mathbb{Z}}$.

Lemma 2.43. *If S,T and T,U form close pairs, then so do S,U.*

Sketch of proof. One has to check that $|\pi_\alpha(S)\cap\pi_\alpha(T)\cap\pi_\alpha(U)|$ grows much faster than the cardinality of its complement in $\pi_\alpha(S)\cup\pi_\alpha(T)\cup\pi_\alpha(U)$. This can be achieved by writing the latter set as a disjoint union of seven parts, according to the various inclusion relations, and then applying the hypotheses and (29). For example, one has

$$\lim_{\alpha\in\mathcal{J}} \frac{|\pi_\alpha(S) - (\pi_\alpha(T)\cup\pi_\alpha(U))|}{|(\pi_\alpha(S)\cap\pi_\alpha(T)) - \pi_\alpha(U)| + |\pi_\alpha(S)\cap\pi_\alpha(T)\cap\pi_\alpha(U)|} = 0$$

because S and T form a close pair. Verifying all details is simple but cumbersome and we leave the task to the reader. $\square$

Lemma 2.43 immediately implies that forming a close pair is an equivalence relation. More interesting for us is the following consequence.

Corollary 2.44. *If S and T form a close pair, so do their closures.*

Proof. By Lemma 2.43 and Example 2.38. $\square$

3. Closed subsets of $\widehat{D}$

3.1. **The ring $\widehat{D}$.** Let F be a global field. We fix a finite set $\mathcal{S}$ of places of F (containing the archimedean ones, if there are any) and let D be the ring of $\mathcal{S}$-integers in F: that is,

$$D := \big\{x \in F \mid v(x) \geqslant 0 \text{ for all } v \notin \mathcal{S}\big\}.$$

Let $\mathcal{I}(D)$ denote the set of all non-zero ideals of D and $\mathcal{P}(D)$ the subset of non-zero prime ideals. We define

$$\widehat{D} := \varprojlim D/\mathfrak{a} \tag{30}$$

where the limit is taken over $\mathfrak{a} \in \mathcal{I}(D)$. Each reduction modulo $\mathfrak{a}$ map $\pi_\mathfrak{a}\colon D \twoheadrightarrow D/\mathfrak{a}$ extends by continuity to a ring homomorphism

$$\hat{\pi}_\mathfrak{a}\colon \widehat{D} \twoheadrightarrow D/\mathfrak{a}\,.$$

By construction there is a canonical injection of D into $\widehat{D}$ and in the following we will always think of D as a (dense) subring of $\widehat{D}$.

For every $\mathfrak{p} \in \mathcal{P}(D)$, we also have the $\mathfrak{p}$-adic completion

$$\widehat{D}_\mathfrak{p} := \varprojlim D/\mathfrak{p}^n.$$

These objects are related by a canonical isomorphism of topological rings

$$\widehat{D} \simeq \prod_{\mathfrak{p}\in\mathcal{P}(D)} \widehat{D}_\mathfrak{p} \tag{31}$$

(a proof can be found in [10, Theorem 2.1]). For simplicity, in the following we shall usually think of (31) as an equality. A consequence is that for every $\mathfrak{p}$ there is a canonical projection $\hat{\pi}_{\mathfrak{p}^\infty}\colon \widehat{D} \to \widehat{D}_\mathfrak{p}$. Each ring $\widehat{D}_\mathfrak{p}$ is endowed with a discrete valuation, which, composing with $\hat{\pi}_{\mathfrak{p}^\infty}$, yields a valuation

$$v_\mathfrak{p}\colon \widehat{D} \longrightarrow \mathbb{N}\cup\{\infty\}$$

which extends the $\mathfrak{p}$-adic valuation on D. Note also that the canonical injection of D into $\widehat{D}_\mathfrak{p}$ factors via the map $\hat{\pi}_{\mathfrak{p}^\infty}$.

By abuse of notation, we shall use the symbols $\hat{\pi}_\bullet$ also when the domain is $\widehat{D}^n$, with $n > 1$.

3.1.1. *Some notations.* The following notations shall be used throughout this paper:

- $\hat{X}$ is the closure of $X \subseteq \widehat{D}^n$
- for I any ideal of $\widehat{D}$, its index is denoted

$$\|I\| := |\widehat{D}/I| \in \mathbb{N} \cup \{\infty\} \tag{32}$$

  and given $a \in \widehat{D}$ we use the shortening $\|a\|$ for $\|a\widehat{D}\|$;
- $\widehat{D}^*$ is the group of units of $\widehat{D}$ (not to be confused with $\widehat{D^*}$, the closure of D^* in $\widehat{D}$).

In the case of an ideal $\mathfrak{a}$ of D, it is easy to check that one has $\hat{\mathfrak{a}} = \mathfrak{a}\widehat{D}$; moreover the equality

$$D/\mathfrak{a} = \widehat{D}/\hat{\mathfrak{a}}$$

holds for every $\mathfrak{a} \in \mathcal{I}(D)$.

3.2. **Eulerian sets.** Following [10, Definition 6.1], we say that $X \subset \widehat{D}^n$ is *Eulerian* if

$$\hat{X} = \prod_{\mathfrak{p}} X(\mathfrak{p}) \,,$$

where $X(\mathfrak{p})$ is the closure of $\hat{\pi}_{\mathfrak{p}^\infty}(X)$ in $\widehat{D}_{\mathfrak{p}}^n$. If moreover each $X(\mathfrak{p})$ is open, we say that X is *openly Eulerian.* For example, coprime pairs in $\mathbb{Z}^2$ are an openly Eulerian subset of $\widehat{\mathbb{Z}}^2$ (this is a special case of [10, Corollary 6.11]).

Remark 3.1. Since each $X(\mathfrak{p})$ is compact, so is their product. Because $\widehat{D}$ is Hausdorff, it follows that the inclusion

$$\hat{X} \subseteq \prod_{\mathfrak{p} \in \mathcal{P}(D)} X(\mathfrak{p}) \tag{33}$$

always holds. The hard part in showing that a set is Eulerian is to prove the opposite inclusion.

Theorem 3.2. *If $X \subseteq \widehat{D}^n$ is be openly Eulerian, then the procounting measure μ_X exists.*

Proof. Since each $X(\mathfrak{p})$ is compact open, Lemma 2.29 yields the existence of $\mu_{X(\mathfrak{p})} \in \mathcal{D}(\widehat{D}_{\mathfrak{p}}, \mathbb{R})$. Now apply Proposition 2.35 to obtain the procounting measure of $\hat{X}$. By Lemma 2.28, this is the same as μ_X . □

Remark 3.3. The results used in the proof of Theorem 3.2 are true for distributions with coefficients in any field of characteristic 0: therefore, if X is openly Eulerian, one has $\mu_X \in \mathcal{D}(X, \mathbb{Q}_p)$ for every prime p. This gives hopes for a possible connection of our theory with p-adic zeta functions. Indeed, taking (say) $X = \widehat{\mathbb{Z}}^*$, the procounting measure is just the Haar measure on $\widehat{\mathbb{Z}}^*$, which is deeply connected with the Riemann zeta function; and the latter famously admits p-adic interpolation.

In the following, we shall say that X is *almost openly Eulerian* if it forms a close pair with an openly Eulerian set.

Corollary 3.4. *If $X \subseteq \widehat{D}^n$ is almost openly Eulerian, then its procounting measure exists.*

Proof. Obvious from Proposition 2.40. □

Lemma 3.5. *Let $S, T \subseteq \widehat{D}$ be openly Eulerian. If they are both closed in $\widehat{D}$, then also $S \cap T$ is openly Eulerian.*

Proof. If S and T are openly Eulerian, the statement reduces to the obvious equalities

$$S \cap T = \Big(\prod S(\mathfrak{p})\Big) \cap \Big(\prod T(\mathfrak{p})\Big) = \prod \big(S(\mathfrak{p}) \cap T(\mathfrak{p})\big).$$

□

Remark 3.6. Some form of openness is necessary: if S and T are only Eulerian, then S' close to S does not imply that $S' \cap T$ is close to $S \cap T$. For an example, take $D = \mathbb{Z}$, $S = \{1\} \cup 2\widehat{\mathbb{Z}}$, $S' = \{3\} \cup 2\widehat{\mathbb{Z}}$ and $T = \{1\}$.

3.2.1. *Prime elements.* Recall that the prime elements of D are those irreducible $x \in D$ such that the ideal xD is prime. Coherently with the notation of [10], we shall denote the set of such elements by $\mathrm{Irr}(D)$.

The closure of $\mathrm{Irr}(D)$ was computed in [10, Theorem 3.2], which proves the equality

$$\widehat{\mathrm{Irr}(D)} = \widehat{D}^* \sqcup \widehat{D^*}\,\mathrm{Irr}(D)\,. \tag{34}$$

Here $\widehat{D^*}$ is the closure of D^* in $\widehat{D}$. Note that $\widehat{D^*}$ is a subgroup of $\widehat{D}^*$ and it is much smaller than the latter: actually, the index is infinite (see [10, Proposition 3.7] for a proof).

Theorem 3.7. *The sets $\widehat{D}^*$ and $\mathrm{Irr}(D)$ form a close pair.*

Proof. Let $\mathfrak{n} \in \mathcal{I}(D)$. By (34) and Lemma 2.1, we obtain

$$\hat{\pi}_{\mathfrak{n}}(\mathrm{Irr}(D)) = (D/\mathfrak{n})^* \sqcup \hat{\pi}_{\mathfrak{n}}(D^*)T_{\mathfrak{n}}\,,$$

where $T_{\mathfrak{n}}$ is a set of non-associated primes of D dividing $\mathfrak{n}$. Since $\hat{\pi}_{\mathfrak{n}}(\widehat{D}^*) = (D/\mathfrak{n})^*$, it suffices, by (29), to prove

$$\lim_{\mathfrak{n}\to 0} \frac{|\hat{\pi}_{\mathfrak{n}}(D^*)|\,|T_{\mathfrak{n}}|}{|(D/\mathfrak{n})^*|} = 0\,. \tag{35}$$

Without loss of generality, we can assume $v_{\mathfrak{p}}(\mathfrak{n}) \geqslant ??$ for every prime ideal $\mathfrak{p}$ dividing $\mathfrak{n}$. By the Chinese remainder theorem, there is a surjective homomorphism

$$\psi_{\mathfrak{n}}\colon (D/\mathfrak{n})^* \twoheadrightarrow \prod_{\mathfrak{p}|\mathfrak{n}} (D_{\mathfrak{p}}/\mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{n})})^*/((D_{\mathfrak{p}}/\mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{n})})^*)^2\,.$$

The assumption $v_{\mathfrak{p}}(\mathfrak{n}) \geqslant ??$ implies that each factor $(D_{\mathfrak{p}}/\mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{n})})^*$ is an abelian group of even order. Hence we obtain

$$|\psi_{\mathfrak{n}}\big((D/\mathfrak{n})^*\big)| \geqslant 2^{\omega(\mathfrak{n})}\,,$$

where $\omega\colon \mathcal{I}(D) \to \mathbb{N}$ is the function which counts how many distinct primes divide an ideal. On the other hand, Dirichlet's unit theorem (in its S-units version) states that D^* is a finitely generated group of rank $|S|-1$ and with cyclic torsion: hence $\psi_{\mathfrak{n}}\big(\hat{\pi}_{\mathfrak{n}}(D^*)\big)$ is at most a product of $|S|$ cyclic groups of order 2 and therefore

$$|\psi_{\mathfrak{n}}\big(\hat{\pi}_{\mathfrak{n}}(D^*)\big)| \leqslant 2^{|S|}\,.$$

Together with the obvious inequality $|T_n| \leqslant \omega(\mathfrak{n})$, this implies

$$\frac{|\hat{\pi}_{\mathfrak{n}}(D^*)|\,|T_{\mathfrak{n}}|}{|(D/\mathfrak{n})^*|} \leqslant \frac{|\psi_{\mathfrak{n}}\big(\hat{\pi}_{\mathfrak{n}}(D^*)\big)|\,\omega(\mathfrak{n})}{|\psi_{\mathfrak{n}}\big((D/\mathfrak{n})^*\big)|} \leqslant \frac{2^{|S|}\,\omega(\mathfrak{n})}{2^{\omega(\mathfrak{n})}}\,.$$

Equality (35) follows by noticing $\lim\limits_{\mathfrak{n}\to 0} \omega(\mathfrak{n}) = +\infty$. ☐

Remark 3.8. The proof above also implies $[\widehat{D}^* : \widehat{D^*}] = \infty$, by an argument avoiding any use of zeta functions (differently from [10, Proposition 3.7]).

Corollary 3.9. *The set $\mathrm{Irr}(D)$ is almost openly Eulerian. The procounting measure $\mu_{\mathrm{Irr}(D)}$ is the Haar measure of the compact group $\widehat{D}^*$.*

Proof. It suffices to observe that $\widehat{D}^*$ is openly eulerian, since (31) implies $\widehat{D}^* = \prod \widehat{D}_{\mathfrak{p}}^*$ and the group of units is compact open in each ring $\widehat{D}_{\mathfrak{p}}$. ☐

3.2.2. *Polynomial images and preimages.* It is straightforward to check that polynomial images are Eulerian.

Proposition 3.10. *For any $f \in D[x_1, ..., x_n]$ and $X \subseteq D^n$, one has:*

- **(a)** *if X is Eulerian, then so is $f(X)$;*
- **(b)** *if $f(X)$ is Eulerian and $\hat{X}$ is the closure of $f^{-1}\big(f(X)\big)$, then X is Eulerian.*

Proof. This is [10, Proposition 6.6]. ☐

Example 3.11. The image of a degree 1 polynomial is an ideal coset. The next simplest example is $f(x) = x^2$, in which case for $D = \mathbb{Z}$ one has

$$f(\mathbb{Z}_p) = \{0\} \sqcup \bigsqcup_{n \in \mathbb{N}} p^{2n} f(\mathbb{Z}_p^*)$$

where

$$f(\mathbb{Z}_p^*) = \begin{cases} 1 + 8\mathbb{Z}_2 & \text{if } p = 2 \\ \boldsymbol{\mu}_{(p-1)/2} \times (1 + p\mathbb{Z}_p) & \text{if } p \neq 2 \end{cases}$$

as discussed in [20, II, §3.3]. (Here $\boldsymbol{\mu}_n$ denotes the group of roots of unity of order n in $\mathbb{Z}_p$.)

Polynomial image can be approximated by openly Eulerian sets.

Lemma 3.12. *For every non-zero polynomial $f \in D[x]$, one has $f(\widehat{D}_{\mathfrak{p}}) = U_{\mathfrak{p}} \sqcup V_{\mathfrak{p}}$, where $U_{\mathfrak{p}}$ is open and $V_{\mathfrak{p}}$ finite.*

Proof. Let f' be the derivative of f and define

$$Z_{\mathfrak{p}} := \{\alpha \in \widehat{D}_{\mathfrak{p}} \mid f'(\alpha) = 0\}.$$

Now we can take $V_{\mathfrak{p}} := f(Z_{\mathfrak{p}})$ and $U_{\mathfrak{p}} := f(\widehat{D}_{\mathfrak{p}}^n) - V_{\mathfrak{p}}$. Indeed, $V_{\mathfrak{p}}$ is finite because so is $\mathbb{Z}_{\mathfrak{p}}$. Given $a_0 = f(\alpha) \in U_{\mathfrak{p}}$, it follows $f'(\alpha) \neq 0$ for some i. Hence, for $|a - a_0|$ small enough we have that α approximates a zero of the polynomial

$$f(\alpha) - a.$$

Thus, by Hensel Lemma, we have $\beta \in \widehat{D}_{\mathfrak{p}}$ such that $f(\beta) = a$. This shows that $U_{\mathfrak{p}}$ is open. □

Corollary 3.13. *For every $f \in D[x_1, ..., x_n]$, one has*

$$f(\widehat{D}) = U \sqcup V$$

where $U = \prod U_{\mathfrak{p}}$ e $\mu(V) = 0$.

Proof. It follows from Lemma 3.12 and the equality $f(\widehat{D}^n) = \prod_{\mathfrak{p}} f(\widehat{D}_{\mathfrak{p}}^n)$ proved in Lemma **??**. □

The set $U \cap D$ is not openly Eulerian since U is not closed. Nonetheless, covering U with an increasing family of open compact $(U_i)_i$, we get that $U_i \cap D$ is openly Eulerian. Consequently, $f(D)$ can be approximated by openly Eulerian sets.

Corollary 3.14. *Let $f\colon D \to D^d$ be a polynomial map. Then, for $X = \widehat{f(D)} \subset \widehat{D}^d$, the distribution μ_X exists.*

Proof. By Lemma **??**, we have that X is quasi-Eulerian; more explicitly

$$\widehat{f(D)} = (\prod_{\mathfrak{p}} Y_{\mathfrak{p}} \cup V_{\mathfrak{p}})$$

where every $V_{\mathfrak{p}}$ is a closed set of measure zero. □

Remark 3.15. Consider the polynomial $x_1^2 + x_2^2 + x_3^2 + x_4^2$. Lagrange's four-squares theorem implies that $f(\mathbb{Z}_p^4) = \mathbb{Z}_p$ for every p, but $f(\mathbb{Z}) = \mathbb{N}$. In this case, the problem clearly lies in the fact that we ignore the Archimedean prime. From the point of view studied in [14] and [10], the issue is that $f(\mathbb{Z})$ is Eulerian, but not closed.

3.3. **Bateman-Horn conjecture.**

Proposition 3.16. *Given $f_1, \dots, f_k \in \widehat{D}[x]$ satisfying the same local hypothesis as in Conjecture 1.1, let $\varphi\colon \widehat{D} \to \widehat{D}^k$ be the map $a \mapsto \big(f_1(a), \dots, f_k(a)\big)$. Then the set $\varphi^{-1}\big((\widehat{D}^*)^k\big)$ admits a procounting measure.*

Proof. We show that $\varphi^{-1}\big((\widehat{D}^*)^k\big)$ is almost openly Eulerian.

For $i \in \{1, \dots, k\}$, consider $Y_i = f_i^{-1}\big(f_i(\widehat{D}) \cap \widehat{D}^*\big)$, so to have

$$\varphi^{-1}\big((\widehat{D}^*)^k\big) = \bigcap_{i=1}^k Y_i\,.$$

By Proposition 3.10.(a) one has

$$f_i(\widehat{D}) \cap \widehat{D}^* = \prod_{\mathfrak{p}} f_i(\widehat{D}_{\mathfrak{p}}) \cap \prod_{\mathfrak{p}} \widehat{D}^*_{\mathfrak{p}} = \prod_{\mathfrak{p}} \big(f_i(\widehat{D}_{\mathfrak{p}}) \cap \widehat{D}^*_{\mathfrak{p}}\big),$$

showing that the hypotheses of Proposition 3.10.(b) apply. Therefore each Y_i is Eulerian and we obtain

$$\varphi^{-1}\big((\widehat{D}^*)^k\big) = \bigcap_{i=1}^{k} \prod_{\mathfrak{p}} Y_i(\mathfrak{p}) = \prod_{\mathfrak{p}} \bigcap_{i=1}^{k} Y_i(\mathfrak{p})\,.$$

In order to conclude the proof, it suffices to check that $\mu_{\widehat{D}_{\mathfrak{p}}}\big(\partial(\cap_i Y_i(\mathfrak{p}))\big) = 0$ holds for every $\mathfrak{p}$. This follows from Lemma 3.12. □

Remark 3.17. $\mu_{\varphi^{-1}((\widehat{\mathbb{Z}}^*)^k)(\mathfrak{p})} =$

Conjecture 3.18. *Under the same hypotheses as in Conjecture 1.1 we have that the following limit exists and*

$$\lim_{n\to 0} \mu_{n,\varphi^{-1}(\mathcal{P}^k)} = \mu_{\varphi^{-1}((\widehat{D}^*)^k)}\,.$$

We now provide some specific examples, testing our conjecture in the case of the famous Theorem of Dirichlet on arithmetic progressions of prime numbers (where $k = \deg(f) = 1$), and also in the case of the twin primes situation, where $k = 2$, $f_1(x) = x$ and $f_2(x) = x + 2$. We will then conclude by proving that it implies that Schinzel's hypothesis is true.

3.3.1. *Case 1: Dirichlet's Theorem.* In its quantitative formulation, Dirichlet's Theorem on arithmetic progressions states that, given $a, b \in \mathbb{N}$, $(a, b) = 1$, then

$$\mathrm{d}_{\mathrm{as},\mathcal{P}}((a + b\mathbb{Z}) \cap \mathcal{P}) = \frac{1}{\varphi(b)}$$

(where $\mathrm{d}_{\mathrm{as},\mathcal{P}}$ denotes the relative density of a subse of $\mathcal{P}$). As anticipated, we are here in a situation where $k = \deg(f) = 1$, where $f(x) = a + bx$. If we apply our profinite reformulation of Bateman-Horn conjecture to this case, we first note that if $m \in \mathbb{Z}$ is such that $a + bm \in \mathcal{P}$, then $m \in b^{-1}\mathcal{P} - \frac{a}{b}$. Then

$$f^{-1}(\widehat{\mathcal{P}}) = \frac{1}{b}\widehat{\mathcal{P}} - \frac{a}{b} = (\frac{1}{b}\widehat{\mathbb{Z}}^* - \frac{a}{b}) \sqcup f^{-1}(\mathcal{P}).$$

For any $n \in \mathbb{N}$ we remark that

$$\hat{\pi}_n(f^{-1}(\widehat{\mathcal{P}})) = f_n^{-1}(\hat{\pi}_n(\widehat{\mathcal{P}}))$$

where $f_n : \widehat{\mathbb{Z}}/n\widehat{\mathbb{Z}} \to \widehat{\mathbb{Z}}/n\widehat{\mathbb{Z}}$ is defined so to have $\hat{\pi}_n \circ f = f_n \circ \hat{\pi}_n$.

Now, we assume that $p \in \mathcal{P}$ is such that $p|b$. In this case we have that

$$f^{-1}_{p^{v_p(n)}}(\hat{\pi}_{p^{v_p(n)}}(\mathcal{P})) = \{x \in \mathbb{Z}/p^{v_p(n)}\mathbb{Z} \ :\ a + bx \in (\mathbb{Z}/p^{v_p(n)}\mathbb{Z})^* \sqcup \{p\}\} = \mathbb{Z}/p^{v_p(n)}\mathbb{Z}$$

where the first equality follows by the profinite formulation of Dirichlet's Theorem on arithmetic progressions:

$$\widehat{\mathcal{P}} = \widehat{\mathbb{Z}}^* \sqcup \mathcal{P}$$

and the second equality follows by our assumption that a and b are coprime, and using the hypothesis that $p|b$.

If, instead, $p \nmid b$, we have that

$$f^{-1}_{p^{v_p(n)}}(\hat{\pi}_{p^{v_p(n)}}(\mathcal{P})) = \{(\mathbb{Z}/p^{v_p(n)}\mathbb{Z})^* - \frac{a}{b}\} \sqcup \{\frac{p-a}{b}\}.$$

For this reason, it follows that

$$\hat{\pi}_n(f^{-1}(\widehat{\mathcal{P}})) = \prod_{p|b} \mathbb{Z}/p^{v_p(n)}\mathbb{Z} \times \prod_{p\nmid b} \{(\mathbb{Z}/p^{v_p(n)}\mathbb{Z})^* - \frac{a}{b}\} \sqcup \{\frac{p-a}{b}\}.$$

In particular, we have that

$$|\hat{\pi}_n(f^{-1}(\widehat{\mathcal{P}}))| = \prod_{p|(n,b)} p^{v_p(n)} \times \prod_{p|n, p\nmid b} (|(\mathbb{Z}/p^{v_p(n)}\mathbb{Z})^*| + 1).$$

Therefore

$$\hat{\pi}_n(f^{-1}(\widehat{\mathcal{P}}))) = \frac{1}{|\hat{\pi}_n(f^{-1}(\widehat{\mathcal{P}}))|} \sum_{x \in \hat{\pi}_n(f^{-1}(\widehat{\mathcal{P}}))} \delta_x = \prod_{p|n} \left(\prod_{p|b} \mu_{\mathbb{Z}_p} \times \prod_{p \nmid b} \mu_{\{\mathbb{Z}_p^* - a/b\}} \right) \times \prod_{p \nmid n} \mu_{\mathbb{Z}_p}$$

as $\mathbb{Z}_p$ and $\mathbb{Z}_p^*$ are all open groups, on which the definition of the corresponding measure is straightforward and coincident with the distributional construction we make in Definition 2.21. We can now easily conclude that

$$\lim_{n \to 0} \mu_{n, f^{-1}(\widehat{\mathcal{P}})} = \prod_{p|b} \mu_{\mathbb{Z}_p} \times \prod_{p \nmid b} \mu_{\mathbb{Z}_p^*}.$$

It is also easy to see that

$$\mu_{f^{-1}(\widehat{\mathbb{Z}}^*)} = \prod_{p|b} \mu_{\mathbb{Z}_p} \times \prod_{p \nmid b} \mu_{\mathbb{Z}_p^* - a/b}.$$

This provides confirmation of Conjecture 1.2 in the specific case of Dirichlet's Theorem.

3.3.2. *Case 2: Twin primes.* The famous Twin Primes Conjecture states in its quantitative formulation that if we take $k = 2$, $f = (f_1, f_2)$, such that $f_1(X) = X$ and $f_2(X) = X + 2$, we then have the following

$$|\{m \in \mathbb{N} \cap [2, x] \ : \ f_1(m) = m \text{ and } f_2(m) = m + 2 \in \mathcal{P}\}| \sim_{x \to +\infty} 2 \prod_{p \geq 3} \frac{p(p-2)}{(p-1)^2} \int_2^x \frac{dt}{(\log(t))^2}.$$

In this specific situation our proposed Conjecture 1.2 would be that

$$\lim_{n \to 0} \mu_{n, f_1^{-1}(\mathcal{P}) \cap f_2^{-1}(\mathcal{P})} = \mu_{f^{-1}((\widehat{\mathbb{Z}}^*)^2)}.$$

While computing the limit above appears extremely hard, we can easily provide a specific description of the right hand side.

In particular, if we express the general element $(x_p)_{p \in \mathcal{P}}$ of $\widehat{\mathbb{Z}}^* = \prod_{p \in \mathcal{P}} \mathbb{Z}_p^*$ as a string with its components labeled by the prime integers, we first remark that

$$x_2 \in \mathbb{Z}_2^* \iff x_2 + 2 \in \mathbb{Z}_2^*$$

easily.

If $p \neq 2$, we then remark that

$$x_p \in f^{-1}((\mathbb{Z}_p^*)^2) \iff x_p \in \mathbb{Z}_p^* \cap (\mathbb{Z}_p^* - 2).$$

Therefore

$$f^{-1}((\widehat{\mathbb{Z}}^*)^2) = \mathbb{Z}_2^* \times \prod_{p \neq 2} (\mathbb{Z}_p^* \cap (\mathbb{Z}_p^* - 2)).$$

As it is not hard to show that if $p \neq 2$, then

$$\mathbb{Z}_p^* \cap (\mathbb{Z}_p^* - 2) = \mathbb{Z}_p^* \setminus 2Ker(\pi_p^*)$$

where

$$\pi_p^* : \mathbb{Z}_p^* \twoheadrightarrow (\mathbb{Z}/p\mathbb{Z})^*$$

it then follows that

$$\mu_{\widehat{\mathbb{Z}}^*}(f^{-1}((\widehat{\mathbb{Z}}^*)^2)) = \prod_{p > 2} \left(1 - \frac{1}{p-1}\right).$$

In particular, the distribution associated to $f^{-1}((\widehat{\mathbb{Z}}^*)^2)$ will be described as

$$\mu_{f^{-1}((\widehat{\mathbb{Z}}^*)^2)} = \mu_{\mathbb{Z}_2^*} \times \prod_{p > 2} \frac{p-2}{p-1} \mu_{\mathbb{Z}_p^{*}*}.$$

3.3.3. *Case 3: Landau's conjecture.* Another relevant case where Conjecture 1.2 can be tested and shows notable similarities with the classic formulation is represented by the famous Landau's conjecture, stating that there are infinitely many primes of the form

$$n^2 + 1$$

with $n \in \mathbb{N}$. In particular, as in the previous case we are able to provide an explicit expression to the measure $\mu_{f^{-1}(\widehat{\mathbb{Z}}^*)}$, where $f(X) = X^2 + 1$.
We start by noting that

$$f^{-1}(\widehat{\mathbb{Z}}^*) = \{x \in \widehat{\mathbb{Z}},\ \text{such that } x^2 + 1 \in \widehat{\mathbb{Z}}^*\} = \prod_{p \in \mathcal{P}} \{x_p \in \mathbb{Z}_p,\ \text{such that } x_p^2 + 1 \in {\mathbb{Z}_p}^*\}.$$

Obviously, $x_2^2 + 1 \in {\mathbb{Z}_2}^*$ if and only if $x_2 \in 2\mathbb{Z}_2$. Also, if $p > 2$, we have that

$${x_p}^2 + 1 \in {\mathbb{Z}_p}^* \iff {x_p}^2 \not\equiv -1 \ mod\ (p)$$

which means that

$$f^{-1}({\mathbb{Z}_p}^*) = \begin{cases} \mathbb{Z}_p \text{ if } p \equiv 3 \ mod\ (4) \\ \mathbb{Z}_p \setminus U_p \text{ if } p \equiv 1 \ mod\ (4), \end{cases}$$

where

$$U_p = (\alpha_p + p\mathbb{Z}_p) \sqcup (\beta_p + p\mathbb{Z}_p) \ \text{ for some } \alpha_p, \beta_p \in \mathbb{Z}_p \text{ such that } {\alpha_p}^2 = {\beta_p}^2 = -1.$$

In particular, we have that

$$f^{-1}(\widehat{\mathbb{Z}}^*) = 2\mathbb{Z}_2 \prod_{p \equiv 3 \ mod\ (4)} \mathbb{Z}_p \prod_{p \equiv 1 \ mod\ (4)} (\mathbb{Z}_p \setminus U_p),$$

hence

$$\mu_{\mathbb{Z}}(f^{-1}(\widehat{\mathbb{Z}}^*)) = \frac{1}{2} \prod_{p \equiv 1 \ mod\ (4)} \left(1 - \frac{2}{p}\right).$$

We now remark that

$$\omega_f(p) = \begin{cases} 1 \text{ if } p = 2 \\ 2 \text{ if } p \equiv 1 \ mod\ (4) \\ 0 \text{ if } p \equiv 3 \ mod\ (4). \end{cases}$$

The Bateman-Horn constant is for this reason as follows:

$$C(f) = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right)^{-1} \left(1 - \frac{\omega_f(p)}{p}\right) = \prod_{p \equiv 1 \ mod\ (4)} \frac{p}{p-1} \left(1 - \frac{2}{p}\right) \prod_{p \equiv 3 \ mod\ (4)} \left(1 - \frac{1}{p}\right)^{-1} =$$

$$= \prod_{p \equiv 1 \ mod\ (4)} \frac{p-2}{p-1} \prod_{p \equiv 3 \ mod\ (4)} \left(1 - \frac{1}{p}\right)^{-1}.$$

We then conclude that

$$\mu_{\widehat{\mathbb{Z}}}(f^{-1}(\widehat{\mathbb{Z}}^*)) = C(f) \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right).$$

Therefore

$$\mu_{f^{-1}(\widehat{\mathbb{Z}}^*)} = C(f) \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right) \mu_{\mathbb{Z}_p}.$$

3.3.4. *Schinzel's hypothesis.* Here we formulate (unfortunately without any idea of how to prove it) a profinite versione of celebrated Schinzel's hypothesis in our profinite setting.

Let $f_1, ..., f_d$ be a collection of irreducible polynomials over $\mathbb{Z}$ with positive leading coefficient and such that there is no prime number p which divides $f_1(n) \cdots f_d(n)$ for all $n \in \mathbb{Z}$, and consider the map $\phi \colon \mathbb{Z} \to \mathbb{Z}^d$ sending $n \mapsto (f_1(n), ..., f_d(n))$. Then, the classical Schinzel's conjecture states that $f_i(n)$'s are all coprime for infinitely many $n \in \mathbb{Z}$. Let f be the product of the f_i's.

Conjecture 3.19 (Profinite H Hypothesis)**.** *If $f(\widehat{\mathbb{Z}}) \cap \widehat{\mathbb{Z}}^* \neq \emptyset$ then the set $\phi(\mathbb{N}) \cap \mathcal{P}^d$ is infinite.*

Proposition 3.20. *Conjecture 3.19 is equivalent to the classical H hypothesis.*

Proof. Fix f as above. Then we notice that $f(\widehat{\mathbb{Z}}) \cap \widehat{\mathbb{Z}}^* \neq \emptyset$ if and only if there exists an open such that $f(\widehat{\mathbb{Z}}) \subseteq \bigcup_{i \leqslant r} p_i \widehat{\mathbb{Z}}$ where the union, by compactness, can be taken finite.

This is equivalent to the existence of a prime p such that $f(\widehat{\mathbb{Z}}) \subseteq p\widehat{\mathbb{Z}}$. If not, by contradiction, for all i there would exist a n_i such that $f(n_i) \notin p\widehat{\mathbb{Z}}$. Thus, since $a = b \bmod p$ implies that $f(a) = f(b) \bmod p$, taking $n = n_i \bmod p$ for all i we would obtain $f(n) \notin \bigcup_{i \leqslant r} p_i \widehat{\mathbb{Z}}$.

Moreover it follows from the density of $\mathbb{Z}$ in $\widehat{\mathbb{Z}}$ and the fact that $p\widehat{\mathbb{Z}}$ is closed that $f(\widehat{\mathbb{Z}}) \subseteq p\widehat{\mathbb{Z}}$ is again equivalent to $f(\mathbb{N}) \subseteq p\mathbb{Z}$.

Therefore $f(\widehat{\mathbb{Z}}) \cap \widehat{\mathbb{Z}}^* \neq \emptyset$ if and only if there is no prime p such that $f(\mathbb{N}) \subseteq p\mathbb{Z}$. □

Proposition 3.21. *Conjecture 1.2 implies Conjecture 3.19.*

Proof. Assume by contradiction that Conjecture 3.19 is false. It would then follow that $\phi^{-1}(\mathcal{P}^d)$ is finite. In particular, for any $n \in \mathbb{N}$ we would have that

$$\mu_{n,\phi^{-1}(\mathcal{P}^d)} = \frac{1}{|\pi_n(\phi^{-1}(\mathcal{P}^d))|} \sum_{x \in \phi^{-1}(\mathcal{P}^d)} \delta_x$$

would be a finite sum of Dirac peaks, and the same would clearly remain true by projecting on the p-adic component $\mathbb{Z}_p$ of $\widehat{\mathbb{Z}}$ (after natural extension of ϕ as a function from $\widehat{\mathbb{Z}}$ to $\widehat{\mathbb{Z}}$. At the same time, as the set $\phi^{-1}(\mathcal{P}^d)$ is quasi-Eulerian, the hypotheses of Theorem **??** hold, and this implies that it is possible to associate a distribution to it, which will be of the form

$$\mu_{\phi^{-1}(\mathcal{P}^d)} = \prod_{p \in \mathcal{P}} \mathbf{1}_{U_p} \, \mu_{\mathbb{Z}_p}$$

where

$$\phi^{-1}(\mathcal{P}^d) = \prod_{p \in \mathcal{P}} U_p \bigsqcup V$$

where U_p are compact open sets and V has Haar measure 0. But if we assume Conjecture 1.2 to be true, it then follows that such a distribution will have to coincide with a finite sum of Dirac peaks, as being the set $\phi^{-1}(\mathcal{P}^d)$ finite, taking the inverse limit for $n \to 0$ of $\mu_{n,\phi^{-1}(\mathcal{P}^d)}$ will give again a finite sum of Dirac peaks on each p-adic component. And this is obviously impossible. □

Appendix A. A numerical experiment

We have tested Conjecture 1.2 in the particular case where $k = 1$ and $\varphi(x) = x + 2$, which turns it into the more specific famous Twin Primes Conjecture, by numerical experiments which seem to confirm it, but involving an unexpectedly high number of twin primes for the relatively small quotients which have been involved. We warmly thank Dr. Alejandro Vidal Lopez (Xi'an Jiaotong - Liverpool University) for the manual set up of a working computer code and his higher knowledge in numerical methods. Let us call

$$S := \varphi^{-1}(\widehat{\mathbb{Z}}^*).$$

If ϕ_n is the projection of $\widehat{\mathbb{Z}}$ onto $\mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}$, we call

$$S_n := \pi_n(S).$$

More specifically

$$S_n = \{[a] \in \mathbb{Z}/n\mathbb{Z} \text{ such that } [a], [a+2] \in (\mathbb{Z}/n\mathbb{Z})^*\}.$$

If Conjecture 1.2 is true, then S_n should be contained in the projection modulo n of all the twin primes, for all $n \in \mathbb{N} - \{0, 1\}$. We have verified this fact for $n = k!$, where $k \leqslant 10$. The software we used is Python and the list of twin primes was generated by the computer as well using a list of the first prime numbers up to 1 billion. This turned out to be necessary already for $k = 8$. We describe the code here below.

```
Code in Python:

import math

number = 10
# bigNumber:
#  denotes number! We consider the group Z/(bigNumber)Z
bigNumber =  math.factorial(number)
# primeDecompPrimes:
#   primes in the prime decomposition of number (not including 2
#   used to calculate the units of Z/(bigNumber)Z

primeDecompPrimes = [3,5,7]



units = []

# Sieve of units.

# We start by adding only odd numbers to the sieve.

# In this way we avoid removing multiples of 2.

for k in range(math.floor(bigNumber/2)):

    units.append(2*k+1)



# We remove all the elements not co-prime with (bigNumber)

# using the information in primeDecompPrimes.

for prime in primeDecompPrimes:

    i = 1

    while i*prime < bigNumber:

        # We try to remove the number i*prime.

        # If the number was removed before, an exception will be launch.

        # In such a case, we catch the exception and ignore it.

        try:

            units.remove(i*prime)
```

```
        except:
            pass
        i += 1

# We write the units of Z/(bigNumber)Z into a file so that
# it does not need to be recalculated.
fileNameUnits ='units_'+str(bigNumber)
fileUnits = open(fileNameUnits, 'w')
for n in units:
    fileUnits.write(str(n)+'\n')
fileUnits.close()

# We calculate the twin units in Z/(bigNumber)Z.
twinUnits = []
for n in range(len(units)-1):
    if units[n+1] - units[n] == 2:
        twinUnits.append(units[n])
if units[-1] == bigNumber -1:
    twinUnits.append(units[-1])

# We save the twin units so that we do not need to calculate them again.
fileNameTwinUnits ='twinUnits_'+str(bigNumber)
fileTwinUnits = open(fileNameTwinUnits, 'w')
for n in twinUnits:
    fileTwinUnits.write(str(n)+'\n')
fileTwinUnits.close()
```

```
# We load the list of twin primes up to 10^9.
# This list was calculated from the list of primes under 10^9,
# generated using the prime_sieve package (see https://pypi.org/project/prime-sieve/)
fileNameTwinPrimes =  'list-of-twinsSieve.txt'
fileTwinPrimes = open(fileNameTwinPrimes, 'r')
twinPrimes =[]
for line in fileTwinPrimes:
    twinPrimes.append(int(line))
fileTwinPrimes.close()


# Finally we calculate the list of twin primes having
# a twin unit of Z/(bigNumber)Z as residue when
# taking modulo (bigNumber).
# We remove those residues from the list of twin units
# and add them to the file finalList_(number).


fileNameFinalList = 'finalList_' + str(number)
fileFinalList = open(fileNameFinalList, 'w')
finalList=[]
for prime in twinPrimes:
    numberToCheck = prime % bigNumber
    if numberToCheck in twinUnits:
        twinUnits.remove(numberToCheck)
        finalList.append(numberToCheck)
        fileFinalList.write(str(numberToCheck)+'\n')
fileFinalList.close()
```

The produced lists of numbers are available at the following link: ————

References

[1] B. Alberts: Counting Functions for Random Objects in a Category. arXiv:2211.07129

[2] S.L. Aletheia-Zomlefer - L. Fukshansky - S.R. Garcia: The Bateman-Horn Conjecture: Heuristics, History, and Applications. *Expo. Math.* **38** (2020), no. 4, 430-479.

[3] L. Bary-Soroker - M. Jarden: On the Bateman-Horn conjecture about polynomial rings. *Münster J. Math.* **5** (2012), 41-58.

[4] L. Bary-Soroker: Hardy-Littlewood tuple conjecture over large finite fields. *Int. Math. Res. Not. IMRN* 2014 (2014), 568-575.

[5] P.T. Bateman - R.A. Horn: A heuristic asymptotic formula concerning the distribution of prime numbers. *Math. Comput.* **16** (1962), 363-367.

[6] A.O. Bender - P. Pollack: On quantitative analogues of the Goldbach and twin prime conjectures over $\mathbb{F}_q[t]$. arXiv:0912.1702

[7] T. Browning - E. Sofos - J. Teräväinen: Bateman-Horn, polynomial Chowla and the Hasse principle with probability 1. arXiv:2212.10373

[8] J. M. Cohen - F. Colonna - D. Singman: Distributions and measures on the boundary of a tree. *J. Math. Anal. Appl.* **293** (2004), no. 1, 89-107.

[9] K. Conrad: Irreducible values of polynomials: a non-analogy. *Number fields and function fields - two parallel worlds*, 71-85, Progr. Math., **239**, Birkhäuser Boston, Boston, MA, 2005.

[10] L. Demangos - I. Longhi: Densities on Dedekind domains, completions and Haar measure. To appear on *Math. Z.*; arXiv:2009.04229

[11] A. Entin: On the Bateman-Horn conjecture for polynomials over large finite fields. *Compos. Math.* **152** (2016), no. 12, 2525-2544.

[12] S.W. Golomb: Arithmetica topologica. In *General Topology and its Relations to Modern Analysis and Algebra*, Proceedings of the symposium held in Prague in September 1961, pp. 179-186. Academia Publishing House of the Czechoslovak Academy of Sciences, Prague, 1962. Persistent URL: http://dml.cz/dmlcz/700933

[13] P. Holdridge: Random Diophantine Equations in the Primes. arXiv:2305.06306

[14] I. Longhi - Y. Mu - F. Saettone: Coset topologies on $\mathbb{Z}$ and arithmetic applications. *Expo. Math.* **41** (2023), no. 1, 71-114.

[15] N. Koblitz: *p-adic numbers, p-adic analysis, and zeta-functions.* Second edition. Graduate Texts in Mathematics, **58**. Springer-Verlag, New York, 1984.

[16] P. Pollack: Simultaneous prime specializations of polynomials over finite fields. *Proc. Lond. Math. Soc. (3)* **97** (2008), 545-567.

[17] B. Poonen: Squarefree values of multivariable polynomials. *Duke Math. J.* **118** (2003), no. 2, 353-373.

[18] W. Rudin: Real and complex analysis. Third edition. *McGraw-Hill Book Co.*, New York, 1987. xiv+416 pp.

[19] W. Rudin: Functional analysis. Second edition. *International Series in Pure and Applied Mathematics. McGraw-Hill, Inc.*, New York, 1991. xviii+424 pp.

[20] J.P. Serre: *A Course in Arithmetic.* Graduate Texts in Mathematics, **7**. Springer-Verlag, New York-Heidelberg, 1973.

[21] A.N. Skorobogatov - E. Sofos: Schinzel Hypothesis on average and rational points. *Invent. Math.* **231** (2023), no. 2, 673-739.

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