SHILDOVSKY LEMMA

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Let

$$L_v y = \dot{y} - Ay = 0, \quad A \in M_{n\times n}(\mathbb{C}(t)), A(0) \neq \infty \tag{1}$$

be a linear system of differential equations with rational coefficients with $t = 0$ being an ordinary point.

Let $P = (p_1, ..., p_n) \in (\mathbb{C}[t]^n)^*$, $\deg P \leq N$, for some $N \gg O_A(1)$ and let $P_i = (L_v^*)^{i-1}P$, $i = 1, ...,$ where $L_v^* P = \dot{P} + PA$. By induction

$$P_i \cdot z = (P \cdot z)^{(i-1)} \tag{2}$$

for any solution $z = z(t)$ of (1).

Let $\mathcal{P}_r$ be a matrix of rational functions whose rows are $P_1, ..., P_r$.

Lemma 0.1 (Shidlovsky)**.** *Assume that*

$$\operatorname{rk} \mathcal{P}_\infty = m \leq n. \tag{3}$$

Then

$$\operatorname{mult}_0 Py \leq Nm + O_A(1) \tag{4}$$

for any solution $y \in \mathbb{C}((t))^n$ of (1) such that $Py \not\equiv 0$.

The simplest case. The simplest particular case of this Lemma is the following familiar one:

Lemma 0.2. *Let*

$$T = \partial^n + a_1(t)\partial^{n-1} + \cdots + a_n(t) \tag{5}$$

be a linear differential operator, and assume that $t = 0$ is its ordinary point. Then any solution $z(t)$ of $Tz = 0$ has a zero of multiplicity at most $n - 1$ at 0.

This follows from the theorem about the existence and uniqueness of solutions of systems of ODEs.

Alternatively, let $z_1, ..., z_n$ be a fundamental system of solutions of $Ty = 0$, and let $W = W(z_1, ..., z_n) = (z_j^{(i-1)})$ be their Wronskian. $\det W$ cannot vanish at an ordinary point, so

$$0 = \operatorname{mult}_0 \det W \geq \operatorname{mult}_0 z_1 - (n - 1) \tag{6}$$

as the first column of W is divisible by $t^{\operatorname{mult}_0 z_1 - (n-1)}$ and the rest is analytic at $t = 0$.

Proof in the case $m = n$. In this case the proof is simpler, so we give it separately to emphasize the main idea.

Write

$$P_{n+1} + a_1(t)P_n + ... + a_n P_1 = 0, \quad a_i \in \mathbb{C}(t).$$

The corresponding linear differential operator (5) has n linearly independent solutions of form $z_j = P \cdot y_j$, where $y_1, ..., y_n$ is a fundamental system of solutions of (1). We have by (2)

$$\det W(z_1, ..., z_n) = \det(z_j^{(i-1)}) = \det\left(\mathcal{P}_n \cdot X\right), \tag{7}$$

where $X = (y_1|...|y_n)$ is the fundamental matrix of (1).

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Clearly $\det X \neq 0$, so

$$\operatorname{mult}_0 \det W = \operatorname{mult}_0 \det \mathcal{P}_n \leq \deg(\det \mathcal{P}_n) = Nn + O_A(1).$$

On the other hand, by (6)

$$\operatorname{mult}_0 \det W \geq \operatorname{mult}_0 z_1 - (n-1) = \operatorname{mult}_0(P \cdot y_1) - (n-1).$$

0.1. **Proof for $m < n$.** Applying L_V^* to

$$P_{m+1} + a_1(t)P_m + ... + a_m P_1 = 0, \quad a_i \in \mathbb{C}(t),$$

one gets

$$P_{m+2} + \widetilde{a}_1(t)P_m + ... + \widetilde{a}_m P_1 = 0.$$

Thus necessarily $m = \operatorname{rk} \mathcal{P}_\infty = \operatorname{rk} \mathcal{P}_m = m$. Denote

$$V(t) = \cap \ker P_i(t) = \cap_{i=1}^m \ker P_i(t), \quad \dim_{\mathbb{C}(t)} V = n - m.$$

The key observation is that

Proposition 0.3. *$V(t)$ is invariant under the flow of (1).*

Indeed, let $z(t)$ be a solution of (1), $z(t_0) \in V(t_0)$ for some ordinary point t_0. Then $(Pz)^k(t_0) = P_k z(t_0) = 0$ for all $k \geq 1$, so $Pz \equiv 0$ and $z(t) \in V(t)$ for all t.

Now, choose solutions $z_1, ..., z_{n-m}$ of (1) such that $\{z_i(t_0)\}$ is a basis of $V(t_0)$ for some t_0 close to 0. Then $z_i(0)$ are still linearly independent, and span

$$V(0) = \lim_{t \to 0} V(t), \quad \dim_{\mathbb{C}} V(0) = n - m.$$

Complete the tuple $z_1, ..., z_{n-m}$ to a basis

$$X(t) = (y_1|...|y_m|z_1|...|z_{n-m})$$

of solutions of (1). Clearly, $\det X(0) \neq 0$. Let

$$X_y = \begin{pmatrix} y_1 & .... & y_m \end{pmatrix}, \quad X_z = \begin{pmatrix} z_1 & .... & z_{n-m} \end{pmatrix},$$

so $X = \begin{pmatrix} X_y & X_z \end{pmatrix}$.

Wronskian. As before,

$$\mathcal{P}_m \cdot X_y = W(Py_1, ..., Py_m),$$

so by (6)

$$\operatorname{mult}_0 \det(\mathcal{P}_m \cdot X_y) \geq \operatorname{mult}_0(Py_1) - (m-1). \tag{8}$$

Covolume. Let $E \in M_{m \times n}(\mathbb{C})$ be an $m \times n$ submatrix of $I_{n \times n}$ such that $\det(EX_z(0)) \neq 0$ and denote $Q = \begin{pmatrix} \mathcal{P}_m \\ E \end{pmatrix}$. Then, as $\mathcal{P}_m X_z = 0$,

$$QX = \begin{pmatrix} \mathcal{P}_m X_y & 0 \\ EX_y & EX_z \end{pmatrix}$$

and therefore

$$\det Q \cdot \det X = \det(QX) = \det(\mathcal{P}_m X_y) \cdot \det(EX_z). \tag{9}$$

Both $\det X$ and $\det EX_z$ are holomorphic and non-vanishing at 0. Thus

$$\operatorname{mult}_0 \det(\mathcal{P}_m X_y) = \operatorname{mult}_0 \det Q \leq \deg \det Q \leq Nm + O_A(1). \tag{10}$$

Comparing (8) and (10) we get (4).