SEMINAR NOTES ON NÉRON MODELS

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Abstract. Notes for a learning seminar at Weizmann. I thank all participants, especially Gal Binyamini and George Papas for the nice discussions. I also thank Zev Rosengarten for his illuminating comments.

1. They are among us

Néron models play a prominently technical role in several celebrated results in arithmetic geometry. On the one hand, Faltings' proof of Mordell conjecture[1] [3] requires the Néron differential on the Néron model of an abelian variety in order to define the Faltings height of such an abelian variety. At a deeper level, his proof also requires Grothendieck's semistable theorem for abelian varieties, which makes an essential use of a result relating the Néron model of the Jacobian of a "nice" curve to the relative Picard scheme of a "nice" integral model of the curve. We mention that the Néron differentials, and consequently the Néron periods, also appear in the Birch–Swinnerton-Dyer conjecture as in [8]. In addition, the arithmetic of cusp modular forms of weight 2 on a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is controlled by the Jacobian of the associated complete modular curve. Such an approach was exploited in [5] and [7] as an important step towards Fermat's last theorem.

2. What is a Néron model?

2.1. **Definition and first properties.** We refer to [2] as the main reference for this section, and we invite to explore Brian Conrad's webpage for more excellent notes. Ler R be a discrete valuation ring, K its fractions field and k its residue field. We point out that assuming the affine setting with R a dvr, rather than S a Dedekind scheme[2], does not imply any loss of content in this theory.

Ideally, the utopian model one dreams about is a proper, smooth R-model. In general, such a model does not exists. In fact, roughly speaking "proper" means that the model is big enough not to miss any points, while "smooth" means small enough to avoid singularities. Intuitively, for a smooth K-scheme X, its Néron model $\mathscr{X}$ is the "best possible" smooth R-scheme extending X, and this essentially happens by forgetting about properness and considering smoothness instead. Withouth any further ado, here is how it is defined.

Definition 2.1. Let X be a smooth, separated K-scheme of finite type. A *Néron model* of X is a smooth, separated, R-scheme $\mathscr{X}$ of finite type such that:

1. $(\mathscr{X}, \varphi)$ is a R-model of X for a fixed $\varphi\colon \mathscr{X}_K \simeq X$;
2. $\mathscr{X}$ satisfies the Néron Mapping Property (NMP), i.e., for every smooth R-scheme $\mathscr{Y}$ and every K-morphism $f_K\colon \mathscr{Y}_K \to \mathscr{X}_K$ there exists a unique R-morphism $f\colon \mathscr{Y} \to \mathscr{X}$ whose pullback under the canonical map $j\colon \operatorname{Spec} K \to \operatorname{Spec} R$ is f_K, namely $f \otimes_R K = f_K$.

Denote, for a commutative unitary ring A, by Sch_A the category of smooth A-schemes. The NMP can be reformulated by saying that the natural map

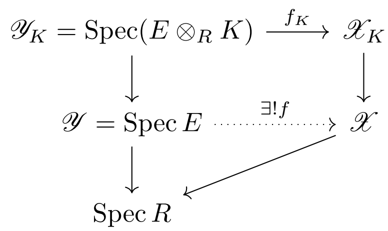
$$\mathrm{Hom}_{Sch_R}(\mathscr{Y}, \mathscr{X}) \to \mathrm{Hom}_{Sch_K}(\mathscr{Y}_K, \mathscr{X}_K)$$

is a bijection. In more functorial terms, the NMP tells us that the pushforward functor $j_*\mathscr{X}_K$ is represented by $\mathscr{X}$, viewed as its functor of points.

[1]A.k.a. Falting's theorem.

[2]I.e., a connected normal Noetherian scheme of dimension 1.

Now, for $\mathscr{Y} = \operatorname{Spec} R$, the NMP implies that every K-point $\operatorname{Spec} K \to \mathscr{X}_K$ extend uniquely to a R-point $\operatorname{Spec} R \to \mathscr{X}$, i.e., $\mathscr{X}(R) = X(K)$. This hints that the NMP is related to the valuative criterion of properness, and in fact it may be viewed as a variant of its to étale local R-morphisms $\operatorname{Spec} E \to \operatorname{Spec} R$. As before, take $\mathscr{Y} = \operatorname{Spec} E$, which is a smooth (since étale is equivalent to smooth and unramified) R-scheme. By the NMP we have



Conversely, if there is such a map f making the diagram commutative, then it extends the point given by f_K.

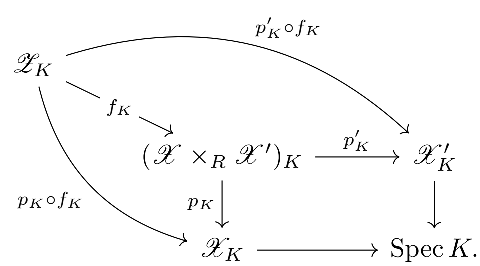
Next result is a manifestation of one of the key features of Néron models: a lot can be done with their existence (which we assume) and the NMP only, without any knowledge of their actual construction.

Lemma 2.2. *Let $\mathscr{X}$ be a smooth, separated R-scheme of finite type which is the Néron model of its generic fiber X, which is a smooth, separated, K-scheme of finite type. Then:*

1. *the formation of Néron models is (functorially) unique: for $\mathscr{N}$ another Néron model of X, there exists a unique isomorphism $\mathscr{X} \to \mathscr{N}$ over R inducing the identity on X;*
2. *the formation of Néron models commutes with the fiber product: let $\mathscr{X}'$ a Néron model of its generic fiber X', with the same hypothesis as above. Then $\mathscr{X} \times_R \mathscr{X}'$ is the Néron model of $X \times_K X'$;*
3. *if X is a K-group scheme, then its group scheme structure extends uniquely to a R-group scheme structure on $\mathscr{X}$;*
4. *the formation of Néron models commutes with étale base change: if R' is an integral domain étale over R with fractions field $K' = R' \otimes_R K$, then $\mathscr{X}_{R'}$ is a Néron model of its generic fiber $\mathscr{X}_{K'}$.*

Proof. (1) Considering $\mathscr{Y} = \mathscr{X}$ and $id_K \colon \mathscr{Y}_K \to X$, it immediately follows from the NMP.

(2) Let $\mathscr{Z}$ be a smooth R-scheme and pick a K-morphism $f_K \colon \mathscr{Z}_K \to (\mathscr{X} \times_R \mathscr{X}')_K$. Consider the following diagram



By the NMP of $\mathscr{X}$ and $\mathscr{X}'$, there exist $p \colon \mathscr{X} \times_R \mathscr{X}' \to \mathscr{X}$, $p' \colon \mathscr{X} \times_R \mathscr{X}' \to \mathscr{X}'$, $p \circ f \colon \mathscr{Z} \to \mathscr{X}$ and $p' \circ f \colon \mathscr{Z} \to \mathscr{X}'$ extending p_K, p'_K, $p_K \circ f_K$ and $p'_K \circ f_K$. Therefore, by the universal property of the fiber product there exists a unique map

$$f \colon \mathscr{Z} \to \mathscr{X} \times_R \mathscr{X}'$$

extending f_K.

(3) We recall that, by the K-group scheme structure of X, there is a K-morphism $m_K \colon X \times_K X \to X$ such that $X(T) \times X(T) \to X(T)$ makes $X(T)$ a group (with the induced law of composition) for

every K-scheme T. By the NMP, we have the following diagram

$$\begin{array}{ccc} X \times X & \xrightarrow{m_K} & X \\ \downarrow & & \downarrow \\ \mathscr{X} \times_R \mathscr{X} & \overset{m}{\dashrightarrow} & \mathscr{X} \end{array}$$

where m extends uniquely m_K. Since $R \to K$ is étale, and étale is stable under base change, we have that $X(T) \simeq \mathscr{X}(T \otimes_K R)$, so that $\mathscr{X}$ inherits the group structure.

(4) Since $\mathscr{X}_{R'}$ is a smooth and separated R'-model of finite type of its generic fiber, we need only to verify the NMP. Let $\mathscr{Y}'$ be a smooth R'-scheme and $\mathscr{Y}'_{K'} \to \mathscr{X}_{K'}$ a K'-morphism. The composition

$$\mathscr{Y}'_{K'} \to \mathscr{X}_{K'} \to \mathscr{X}_K$$

gives a K-morphism extending to a R-morphism $\mathscr{Y}' \to \mathscr{X}$, since $\mathscr{X}$ is the Néron model of X_K. Note that, since $R' \to R$ is étale, that is, smooth and unramified, and $\mathscr{Y}'$ is smooth over R', then $\mathscr{Y}'$ is smooth over R as well. Therefore $\mathscr{Y} \to \mathscr{X}$ yields a R'-morphism $\mathscr{Y}' \to \mathscr{X} \times_R R'$. By the Pullback Lemma applied to

$$\begin{array}{ccccc} \mathscr{Y}'_{K'} & \longrightarrow & \mathscr{X}_{K'} & \longrightarrow & \mathscr{X}_K \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{Y}' & \longrightarrow & \mathscr{X}_{R'} & \longrightarrow & \mathscr{X} \end{array}$$

this is an extension of $\mathscr{Y}'_{K'} \to \mathscr{X}_{K'}$. The uniqueness of such extension follows from the uniqueness of $\mathscr{Y}' \to \mathscr{X}$ and from the NMP of $\mathscr{X}_{R'}$.

□

It is important to remark that Néron models have weak functorial properties: for instance, their formation do *not* commute with ramified base change. Moreover, it also behaves poorly with exact sequences.

Before sketching how to construct a Néron model, next result gives a first property of how isogenies and Néron models interact.

Let us recall that an isogeny f has a *degree* $\deg(f)$, defined as the rank of the finite group scheme $\ker(f)$. We also say that a smooth, connected commutative K-group scheme G of finite type is *semi-abelian* if it is an (1-)extension of an abelian variety by a torus. For G over a general base scheme S, we say that G is semi-abelian if all its fibers are semi-abelian.

Proposition 2.3. *Let G, G' two smooth, commutative, connected K-group scheme of finite type admitting Néron models $\mathscr{G}$ and $\mathscr{G}'$ over R respectively. Let also $f_K : G \to G'$ be an isogeny such that char$(k) \nmid \deg(f_K)$ or that G is semi-abelian. Then there is an isogeny $f : \mathscr{G} \to \mathscr{G}'$ extending f_K. Moreover, there exists an isogeny $g : \mathscr{G}' \to \mathscr{G}$ such that $g \circ f = [\deg(f_K)]$.*

Sketch of proof. By [2, Lemma 2.7.2] we have the following two facts. If G is semi-abelian, then $[\deg(f_K)]$ is finite and flat. On the other hand, for $\mathrm{char}(k) \nmid \deg(f_K)$ we have that $[\deg(f_K)]$ is étale.

Now, since $\ker(f_K) \subset \ker([\deg(f_K)])$, and f_K is flat and onto, we have $G' = G/\ker(f_K)$ and the morphism $G \to G' \to G/\ker([\deg(f_K)])$. Since $[\deg(f_K)]$ is finite and it factors through $G/[\deg(f_K)]$, we have the existence of $g_K : G' \to G$ such that $g_K \circ f_K = [\deg(f_K)]$. By the NMP of $\mathscr{G}$ and $\mathscr{G}'$, we have that f_K and g_K extend to two R-morphisms $f : \mathscr{G} \to \mathscr{G}'$ and $g : \mathscr{G}' \to \mathscr{G}$, such that $g \circ f = [\deg(f_K)]$. Then by our assumptions and the aforementioned facts, we have that $[\deg(f_K)]$ is an isogeny and so are f and g. □

2.2. **An idea of existence.** First of all, we present the local main existence theorem for abelian varieties.

Theorem 2.4. *Let R be a dvr with fractions field K. Let A be an abelian variety over K. Then A admits a Néron model $\mathscr{A}$ over R.*

We will sketch the main steps of its proof following [2]. For an alternative approach via formal groups and invariant differential operators see [4].

Let us begin this highly technical section with some technical definitions.

We recall that a local ring $(R, \mathfrak{m})$ with residue field k is *Henselian* if the Hensel Lemma holds. Namely, for every monic polynomial $f \in R[x]$ and every simple zero α of $f \bmod \mathfrak{m}$, there is a unique $\alpha' \in R$ such that it lifts α and it is a zero of f. If R is Henselian and k is separably closed, then R is called *strictly Henselian*.

Given any local ring R, the process of strict Henselisation gives the smallest strict Henselian local ring containing R. More precisely, let k^s denote a fixed separable closure[3] of k. A *strict Henselisation* of R consists of the pair $(R^{\text{sh}}, i\colon R \to R^{\text{sh}})$ where R^{sh} is a strict Henselian local ring with residue field (isomorphic to) k^s and i is a local morphism with the following universal property. For any strict Henselian local ring A and any local morphism $R \to A$, and a k-embedding of k^s into the residue field of A, there is a unique local morphism $R^{\text{sh}} \to A$ such that the diagram

$$\begin{array}{ccc} A & & \\ \uparrow & \nwarrow & \\ R & \longrightarrow & R^{\text{sh}} \\ \downarrow & & \downarrow \\ k & \hookrightarrow & k^s \end{array}$$

commutes.

Given a scheme Z and an étale morphism $Z \to \operatorname{Spec} R$, we have that R^{sh} can also be constructed as the colimit of $\mathscr{O}_{Z,z}$, for z a point of Z lying above the closed point of $\operatorname{Spec} R$. This hints that the strict Henselisation of $\mathscr{O}_{Z,z}$ is a local ring for the étale topology on Z. In particular, this construction implies that $\mathfrak{m}$ generates the maximal ideal of R^{sh}, and this shows that the latter is again a dvr. Naturally, we denote by K^{sh} its fractions field.

We will soon need the two following technical results.

Lemma 2.5 (EGA IV, 8.8.2)**.** *Let S be a Dedekind base scheme with $s \in S$.*

1. *Let also X and Y be finitely presented S-schemes. Then we have a bijection*
$$\varinjlim_{U \ni s} \operatorname{Hom}_U(X \times_S U, Y \times_S U) \simeq \operatorname{Hom}_{\mathscr{O}_{S,s}}(X \times_S \mathscr{O}_{S,s}, Y \times_S \mathscr{O}_{S,s})$$
where the colimit runs over all open neighborhoods U of s;
2. *let $X_{(s)}$ be an finitely presented $\mathscr{O}_{S,s}$-scheme. Then there are an open neighborhood $S' \subset S$ of s and a S'-scheme X of finite presentation such that $X' \otimes_{S'} \mathscr{O}_{S,s} \simeq X_{(s)}$.*

To introduce the second technical Lemma, we need a relative notion of rational map. Let again S denote a Dedekind base scheme. An open subscheme U of a reduced S-scheme X is *U-dense* if for every closed points $s \in S$, one has that $U_s = U \times_S k(s)$ is Zariski dense in X_s. Let Y be a separated S-scheme. A *S-rational* map $f\colon X \dashrightarrow Y$ is an equivalence class of S-morphisms $U \to Y$ where U is S-dense in X. Lastly, we denote by $\operatorname{dom}(f) = \{x \in X : \text{f is defined at x}\}$ the domain of definition of f, i.e., the largest open subset U on which f is defined.

Lemma 2.6 (Weil Extension Lemma)**.** *Let S be a Dedekind scheme. Let also Z be a smooth S-scheme and G be a smooth, separated S-group scheme. If a S-rational map $f\colon Z \dashrightarrow G$ is defined in codimension $\leqslant 1$, then f is defined everywhere.*

[3]When k is finite, then the separable closure coincides with the algebraic one.

2.2.1. *Step 1: construction of a proper R-model.* Let us recall the notion of schematic closure. Let $\mathscr{X}$ be a flat R-scheme, and Y a closed subscheme of $\mathscr{X}_K$. The *schematic closure* $\mathscr{Y}$ of Y in $\mathscr{X}$ is defined as follows. For an open affine $U = \operatorname{Spec} B$ of $\mathscr{X}$, locally $Y \cap U$ is given by an ideal $\mathfrak{b}$ of $B \otimes_R K$. By taking the pullback of $\mathfrak{b}$ via $j\colon B \to B \otimes_R K$ we define $\mathscr{Y} \cap U$. Note that since $B/j^*\mathfrak{b}$ injects into the K-vector space $B \otimes_R K/\mathfrak{b}$ it follows that $B/j^*\mathfrak{b}$ is a torsion-free R-module. Over a Dedekind domain this condition is equivalent to flatness, and so we have that the schematic closure is flat over R.

As A is projective[4], it is enough to fix a projective embedding of A

$$A \hookrightarrow \mathbb{P}^n_K \hookrightarrow \mathbb{P}^n_R$$

and then take the schematic closure of A in $\mathbb{P}^n_R$, which we denote by $\mathscr{A}_0$. This gives us a proper, flat R-model which is not necessarily smooth.

2.2.2. *Step 2: the smoothening process.* The goal of this step is to obtain a R-morphism $f\colon \mathscr{A}_1 \to \mathscr{A}_0$ such that $f_K\colon \mathscr{A}_{0,K} \to \mathscr{A}_{1,K}$ is an isomorphism and that the canonical map $\mathscr{A}_1^{\mathrm{sm}}(R^{\mathrm{sh}}) \to \mathscr{A}_0(R^{\mathrm{sh}})$ is bijective.

Let $\Omega^1_{\mathscr{X}/R}$ be the sheaf of Kähler differentials of $\mathscr{X}$ over R. For $a\colon \operatorname{Spec} R^{\mathrm{sh}} \to \mathscr{X}$, the R^{sh}-module $a^*\Omega^1_{\mathscr{X}/R}$ decomposes into a free and a torsion part, thanks to the structure theorem for finitely generated modules over a PID, as we mentioned that R^{sh} is a dvr.
The *(Néron measure of) defect of smoothness* $\delta(a)$ at a is the lenght of the torsion part of the R^{sh}-module $a^*\Omega^1_{\mathscr{X}/R}$. The finiteness of $\delta(a)$ follows from the fact that it is a finitely generated torsion module.
This is one of the key ideas of the smoothening process, and next Lemma shows that δ indeed is a reasonable way to measure how $\mathscr{X}$ is far from being smooth.

Lemma 2.7. *Let $\mathscr{X}$ be a R-scheme of finite type such that $\mathscr{X}_K$ is smooth. Consider $\eta \in \mathscr{X}_K$ and $z \in \mathscr{X}_k := \mathscr{X} \times_R \operatorname{Spec} k$ such that $z \in \overline{\{\eta\}}$. In addition, suppose that $\mathscr{X}_K$ is smooth at y of relative dimension $d = \dim_{k(z)} \Omega_{\mathscr{X}/R} \otimes_{\mathscr{O}_{\mathscr{X},z}} k(z)$. Then:*

1. *$\mathscr{X}$ is smooth at z of relative dimension d;*
2. *$a^*\Omega^1_{\mathscr{X}/R}$ is free if and only if the image of a is in $\mathscr{X}^{sm}$;*
3. *the defect of smoothness δ has a maximum on $\mathscr{X}(R^{sh})$.*

Sketch of proof. (1) Let us recall the (non-trivial) fact that a morphism $f\colon \mathscr{X} \to \operatorname{Spec} R$ is smooth of relative dimension d if f is flat and the fibers are geometrically regular of equidimension d. For a morphism of finite type $f\colon \mathscr{X} \to \operatorname{Spec} R$ the set $\{x \in \mathscr{X} : \dim_x \mathscr{X}_{f(x)} \geqslant n\}$ is closed for every $n \geqslant 0$, since by Chevalley's theorem the dimension of the fibers of f is upper-semicontinuous on $\mathscr{X}$. Since $z \in \overline{\{\eta\}}$, then we have that $\dim_z \mathscr{X}_k \geqslant d$. Moreover, the $k(z)$-dimension of $\Omega^1_{\mathscr{X}/R} \otimes k(z) = \Omega^1_{\mathscr{X}_k/k} \otimes k(z)$ is d. We omit the proof of flatness.

(2) By smoothness, $\Omega^1_{\mathscr{X}/R}$ is locally free on an open subset containing the image of a, so that $a^*\Omega^1_{\mathscr{X}/R}$ is free.
On the other hand, let be z and η the images of the special and generic points of $\operatorname{Spec} R^{\mathrm{sh}}$ under a. We have that $\operatorname{rk} a^*\Omega^1_{\mathscr{X}/R} = \dim_z X_K$, so by part (1) we conclude.

(3) We refer to [2, Proposition 6 p.66].

□

Next Lemma shows that the defect of smoothness decreases after "wise" blowing-ups of $\mathscr{X}$. Before stating it, we need one more technical definition.

Let $E \subseteq \mathscr{X}(R^{\mathrm{sh}})$ and let $\mathscr{Y}_k$ be a closed subscheme of $\mathscr{X} \times_R \operatorname{Spec} k$. Let U_k be the largest open subscheme of $\mathscr{Y}_k$ which is smooth over k and $\Omega^1_{\mathscr{X}/R}|_{U_k}$ is locally free. We have that this is dense in $\mathscr{Y}_k$. Then $\mathscr{Y}_k$ is *E-permissible* if the images of the specialisations of the points of E that specialise to k^s-points of Y_k form a (schematically) dense subset of $\mathscr{Y}_k$ contained in U_k.

[4] As a nice consequence of the theorem of the square.

Lemma 2.8. *Let $\mathscr{Y}_k$ be a E-permissible closed subscheme of $\mathscr{X} \times_R \operatorname{Spec} k$. Let $\widetilde{\mathscr{X}} \to \mathscr{X}$ be the blowing-up of $\mathscr{X}$ at $\mathscr{Y}_k$. Let $a \in E$ and denote by $\widetilde{a}$ its unique lifting to a R^{sh}-point of $\widetilde{\mathscr{X}}$. We have that:*

(1) $\delta(\widetilde{a}) \leqslant \max\{0, \delta(a) - 1\}$ *if a specialises to a point of* $\mathscr{Y}_k$*;*
(2) $\delta(\widetilde{a}) = \delta(a)$ *otherwise.*

Armed with this result, we can know introduce the main theorem of this section.

A *smoothening* of $\mathscr{X}$ is a proper R-morphism $f: \mathscr{X}_1 \to \mathscr{X}$ which is an isomorphism on generic fibers and such that $\mathscr{X}_1^{\mathrm{sm}}(R^{\mathrm{sh}}) = \mathscr{X}(R^{\mathrm{sh}})$.

Theorem 2.9 (Smoothening Process)**.** *Let $\mathscr{X}$ be a R-scheme of finite type whose generic fiber X is a smooth K-scheme. Then X admits a smoothening defined as a finite sequence of blowing-ups centered in the singular loci of the successive special fibers.*

Sketch of proof. We just give a (rather coarse[5]) idea of the actual proof. Let $E \subset \mathscr{X}(R^{\mathrm{sh}})$ such that its points specialise into the singular locus of $\mathscr{X}$. Consider the strict decreasing filtration $E = E_1 \supset E_2 \supset \dots$ given by:

- $\mathscr{Y}_i =$ the closure of $\{im(a_k\colon \operatorname{Spec} k^s \to \mathscr{X})|a_k \in E_i\}$;
- $U_i =$ largest open subscheme of $\mathscr{Y}_i$ such that U_i is k-smooth and $\Omega^1_{\mathscr{X}/R}|_{U_i}$ is locally free;
- $E_i =$ points in E_{i-1} specialising into $\mathscr{Y}_{i-1} - U_{i-1}$.

We thus have that $\mathscr{Y}_i$ is $(E_i - E_{i-1})$-permissible. Since the $\mathscr{Y}_i$'s form a decreasing sequence of closed subsets in $\mathscr{X}_k$ which is is Noetherian, there is $n \geqslant 0$ such that $E_{n+1} = \emptyset$. If $n = 0$, we conclude. Otherwise consider the defect of smoothness along E^n

$$\delta(E^n) := \max\{\delta(a)|a \in E^n\}.$$

By Lemma 2.8 we have that the blowing-up $\widetilde{\mathscr{X}} \to \mathscr{X}$ of $\mathscr{Y}_n$ in $\mathscr{X}$ is such that $\delta(\widetilde{a}) < \delta(E^n)$ for every $\widetilde{a} \in \widetilde{E}^n \subset \widetilde{\mathscr{X}}(R^{\mathrm{sh}})$. As above, we construct a new sequence $\widetilde{E}^{n,1} \supset \widetilde{E}^{n,2} \supset \cdots \supset \widetilde{E}^{n,m+1} = \emptyset$ and again either $m = 0$ or we blow up the right closed. We proceed recursively, and after finitely many steps we obtain $\mathscr{X}_1 \to \mathscr{X}$ which is constructed by successive blowing-ups in the singular loci of the special fibers. By construction, we have that every point in the lift of E to $\mathscr{X}_1(R^{\mathrm{sh}})$ factor through $\mathscr{X}_1^{\mathrm{sm}}$. By definition, $\mathscr{X}_1 \simeq \mathscr{X}^{\mathrm{sm}}$, so that all of $\mathscr{X}_1(R^{\mathrm{sh}})$ factor through $\mathscr{X}_1^{\mathrm{sm}}$. □

Thus, if we remove from $\mathscr{X}_1$ its singular locus, we obtain a smooth R-scheme $\widetilde{\mathscr{X}_1}$ of finite type and a R-morphism $\widetilde{\mathscr{X}_1} \to \mathscr{X}_1$ which is an isomorphism on generic fibers and such that $\widetilde{\mathscr{X}_1}(R^{\mathrm{sh}}) = \mathscr{X}_1(R^{\mathrm{sh}})$. Such R-schemes need not to be neither unique nor proper.

2.2.3. *Step 3: the Weak Néron Model.* Let X be a smooth, separated K-scheme of finite type. A *weak* Néron model $\mathscr{X}^{\mathrm{wk}}$ of X is a smooth, separated R-scheme of finite type which is a R-model of X and such that the canonical map $\mathscr{X}^{\mathrm{wk}}(R^{\mathrm{sh}}) \to X(K^{\mathrm{sh}})$ is bijective.

Moreover, a weak Néron model $\mathscr{X}^{\mathrm{wk}}$ of X satifies the Weak Néron Mapping Property (WNMP) if, for any smooth R-scheme $\mathscr{Y}$ with irreducible special fiber $\mathscr{Y}_k$ and any K-rational map $f_K\colon \mathscr{Y}_K \dashrightarrow X$ there is a R-rational map $f\colon \mathscr{Y} \dashrightarrow \mathscr{X}^{\mathrm{wk}}$ extending f_K.

Proposition 2.10. *Let $\mathscr{X}^{wk}$ be a weak Néron model of X. Then $\mathscr{X}^{wk}$ satisfies the WNMP.*

Sketch of proof. Let $\mathscr{Y}$ and f_K be as above. Consider an open dense subscheme $U \subset \mathscr{Y}_K$ upon which f_K is defined. Take now its complement $Z := \mathscr{Y}_K - U$ and denote by $\mathscr{Z}$ the schematic closure of Z in $\mathscr{Y}$. As we showed, $\mathscr{Z}$ is flat over R so that $\dim \mathscr{Z}_k = \dim Z < \dim \mathscr{Y}_K = \dim \mathscr{Y}_k$. Hence we have that $\mathscr{Y}_k - \mathscr{Z}_k$ is Zariski-dense in $\mathscr{Y}_k$, and so $\mathcal{U} = \mathscr{Y} - \mathscr{Z}$ is dense in $\mathscr{Y}$. We thus need to find a rational map $\mathcal{U} \dashrightarrow \mathscr{X}$ extending the morphism $f_K\colon U \dashrightarrow X$. We can now replace $\mathscr{Y}$ with $\mathcal{U}$ so that we have $f_K\colon \mathscr{Y}_K \to X$. By the separatedness of $\mathscr{X}$, we can work locally on $\mathscr{Y}$ and then glue the R-rational maps. In fact, for a R-rational map $\mathscr{Y} \dashrightarrow \mathscr{X}$, two of its representatives necessarily agree on an open dense subset of the intersection of their domains. As $\mathscr{X}$ is separated, they need to agree on the entire intersection. In this

[5]I'm planning to write a clearer version of it hopefully soon.

way they can be glued so to form a unique R-morphism. Assume now that $\mathscr{Y}$ is of finite type. Consider the graph of f_K and denote by T its schematic closure in $\mathscr{Y} \times_R \mathscr{X}$. Denote also by p_1 and p_2 the two projections on the first and second factor respectively. We claim that p_1 is invertible on a dense open of $\mathscr{Y}$. Then the map $p_2 \circ p_1^{-1} \colon \mathscr{Y} \dashrightarrow \mathscr{X}$ is what we have been looking for. Let us show that $p_1(T)_k$ contains a dense open subset of $\mathscr{Y}_k$. We can assume that $R = R^{\mathrm{sh}}$ and that $k = k^s$. Then the set of rational points of $\mathscr{Y}_k$ is dense in $\mathscr{Y}_k$ by [2, Corollary 2.2.13]. Since R is strictly Henselian, and $\mathscr{Y}$ is smooth, we have that $y_k \in \mathscr{Y}_k(k)$ lifts to $y \in \mathscr{Y}(R)$. Denote by x_K the point $f(y_K) \in X(K)$. By definition of weak Néron model, we have that x_K extends to $x \in \mathscr{X}(R)$. Therefore $(y, x) \in T$, so that $y_k \in p_1(T)_k$. As $\mathscr{Y}_k(k) \subset p_1(T)_k$, we have that $p_1(T)_k$ is dense in $\mathscr{Y}_k$. By a theorem of Chevalley the image of a constructible is constructible (here we need the assumption that $\mathscr{Y}$ is of finite type). Thus $p_1(T)_k$ is constructible, so that it contains a non-empty open of $\mathscr{Y}_k$. □

Now, we take the smooth (albeit non-proper) $\mathscr{A}^{\mathrm{wk}} = \mathscr{A}_1^{\mathrm{sm}}$, which, by the smoothening process, is a weak Néron model of A, and hence by the previous Proposition satisfies the WNMP.

2.2.4. *Step 4: birational group laws.* For a very nice exposition involving algebraic spaces, we invite the reader to go through [1].

We begin with a notion which can be viewed as a weak version of a R-group scheme. Let $\mathscr{X}$ be a smooth, separated, faithfully flat[6] R-model of finite type of a smooth K-group scheme X of finite type with multiplication given by $m_K \colon X \times_K X \to X$. A *$R$-birational group law* on $\mathscr{X}$ is a R-rational map $m_R \colon \mathscr{X} \times_R \mathscr{X} \dashrightarrow \mathscr{X}$ extending m_K such that m_R is associative, as far as the rational maps are defined, and the two *universal translations* $\mathscr{X} \times_R \mathscr{X} \dashrightarrow \mathscr{X} \times_R \mathscr{X}$ mapping (x, y) to (x, xy) and (xy, y) respectively are R-birational maps. A *solution* of a R-birational group law m_R on $\mathscr{X}$ is a smooth, separated R-group scheme of finite type $\bar{\mathscr{X}}$ with multiplication $\bar{m}_R$ together with a R-dense open $\mathscr{X}' \subset \mathscr{X}$ and a R-dense open immersion $\mathscr{X}' \hookrightarrow \bar{\mathscr{X}}$ such that $\bar{m}_R$ restricts to $m_R|_{\mathscr{X}'}$. This means that a solution of a birational group law expands a R-dense open to a R-group scheme. Moreover, if a solution exists, then it is unique up to isomorphism.

Next result, originally due to Weil (whose original motivation was the algebraic construction of the Jacobian variety) and then generalized by Artin, is particularly hard (and long) and we will not sketch any proof.

Theorem 2.11. *Let $\mathscr{X}$ and m_R be as above. Then there exists a solution of m_R, and $\mathscr{X} = \mathscr{X}'$ is a R-dense open subscheme of a R-group $\bar{\mathscr{X}}$ with $\bar{m}_R$ restricting to m_R on $\mathscr{X}$.*

Note that the previous result deals only with the case of a dvr as base scheme. For a more general base, one needs to apply the technique of faithfully flat descent.

Let us now go back to the case of an abelian variety A over K and its Néron model $\mathscr{A}$, provided it exists. The sheaf of Kähler differentials $\Omega^d_{\mathscr{A}/R} = \bigwedge^d \Omega^1_{\mathscr{A}/R}$ is a line bundle generated by a (bi-)invariant differential form. In particular, it can be showed that on A there is an invariant global section ω unique up to a constant in $K^\times$.

Let η be the generic point of the special fiber $\mathscr{A}_k$. Then its local ring $\mathscr{O}_{\mathscr{A},\eta}$ is a dvr with maximal ideal generated by the uniformizer π of R and the free $\mathscr{O}_{\mathscr{A},\eta}$-module $\Omega^d_{\mathscr{A}/R,\eta}$ has rank 1, generated by $\pi^{-\operatorname{ord}_\eta(\omega)}\omega$. In this way ω can be viewed as a differential form on $\mathscr{A}$.

Let us label as $\{C_i\}_{i \in I}$ the components of $\mathscr{A} \times \operatorname{Spec} k$, each of them with generic points η_i's. If $\operatorname{ord}_{\eta_i}(\omega) = \min\{\operatorname{ord}_{\eta_i}(\omega) : i \in I\}$, the C_i is called *ω-minimal*. We also denote by $\mathscr{A}_i$ the open subscheme of $\mathscr{A}$ where all components of its special fiber were removed besides C_i. Moreover, we say that $\mathscr{A}_i$ and $\mathscr{A}_j$ are *equivalent* if there is a R-birational map $\mathscr{A}_i \dashrightarrow \mathscr{A}_j$ which restricts to the identity on the (common) generic fiber G. Lastly, since this is an equivalence relation, let us index by I_0 the $\mathscr{A}_i$'s with ω-minimal special fiber, and by I_1 a set of equivalence class representative for the $\mathscr{A}_i$'s with ω-minimal special fiber.

[6]I.e., $\mathscr{X}$ has non-empty fibers over R.

Proposition 2.12. *Let $\mathscr{A}^{bg}$ be the subscheme of $\mathscr{A}$ obtained by removing all the components C_i for $i \in I - I_1$ from the special fiber. Then $\mathscr{A}^{bg}$ has a R-birational group law m_R that extends the group law m_K on G.*

Sketch of proof. By the WNMP of $\mathscr{A}^{\mathrm{wk}}$ the K-morphism m_K extends to a R-rational map

$$m \colon \mathscr{A}^{\mathrm{bg}} \times \mathscr{A}^{\mathrm{wk}} \dashrightarrow \mathscr{A}^{\mathrm{wk}}.$$

Our goal is to show that m induces a R-rational map

$$\mathscr{A}^{\mathrm{bg}} \times \mathscr{A}^{\mathrm{bg}} \dashrightarrow \mathscr{A}^{\mathrm{bg}}.$$

We claim, without any proof, that ω extends to a global section of $\Omega^d_{\mathscr{A}^{\mathrm{wk}}/R}$ and its restriction to $\mathscr{A}^{\mathrm{bg}}$ is a global generator of $\Omega^d_{\mathscr{A}^{\mathrm{bg}}/R}$.
Consider the restriction to the domain of definition of m of the universal left translation $f \colon dom(m) \to \mathscr{A}^{\mathrm{bg}} \times \mathscr{A}^{\mathrm{wk}}$. As f is a $\mathscr{A}^{\mathrm{bg}}$-morphism of left translation, we have that $(p_2 \circ f)^*\omega = \omega|_{dom(m)}$ holds on the generic fiber (and hence everywhere by density), where $p_2 \colon \mathscr{A}^{\mathrm{bg}} \times \mathscr{A}^{\mathrm{wk}} \to \mathscr{A}^{\mathrm{wk}}$. Let ξ be the generic point of the special fiber of $\mathscr{A}^{\mathrm{bg}} \times \mathscr{A}^{\mathrm{wk}}$, and $\zeta := f(\xi)$. We have that $\omega|_{dom(m)}$ is a generator of $\Omega^d_{dom(m)/\mathscr{A}^{\mathrm{bg}}}$ at ξ and $\pi^{-\operatorname{ord}_\zeta(\omega)}\omega|_{dom(m)}$ is a generator of $\Omega^d_{\mathscr{A}^{\mathrm{bg}} \times \mathscr{A}^{\mathrm{wk}}/\mathscr{A}^{\mathrm{bg}}}$ at ζ. Thus

$$(p_2 \circ f)^*\pi^{-\operatorname{ord}_\zeta(\omega)}\omega = \pi^{-\operatorname{ord}_\zeta(\omega)}p_2^*\omega = g\omega|_{dom(m)}$$

for g in the stalk at ξ. Hence $\operatorname{ord}_\zeta(\omega) = 0$ and so $\zeta \in \mathscr{A}^{\mathrm{bg}} \times \mathscr{A}^{\mathrm{bg}}$. Therefore the set of irreducible components of the special fiber of $\mathscr{A}^{\mathrm{bg}} \times \mathscr{A}^{\mathrm{bg}}$ is mapped into itself by f. So we get a morphism $f^{\mathrm{bg}} \colon dom(m) \cap \mathscr{A}^{\mathrm{bg}} \times \mathscr{A}^{\mathrm{bg}} \to \mathscr{A}^{\mathrm{bg}} \times \mathscr{A}^{\mathrm{bg}}$ and the desired R-rational map

$$p_2 \circ f^{\mathrm{bg}} \colon dom(m) \cap \mathscr{A}^{\mathrm{bg}} \times \mathscr{A}^{\mathrm{bg}} \dashrightarrow \mathscr{A}^{\mathrm{bg}}.$$

Similarly one can define universal translations first on $dom(m)$ and then on $dom(m) \cap \mathscr{A}^{\mathrm{bg}} \times \mathscr{A}^{\mathrm{bg}}$, and this show that those induce an isomorphism of $dom(m) \cap \mathscr{A}^{\mathrm{bg}} \times \mathscr{A}^{\mathrm{bg}}$ onto a R-dense subscheme. □

We eventually obtained a smooth, separated R-model $\mathscr{A}^{\mathrm{bg}}$ of finite type of A with a R-birational group law extending the group law of A. By theorem 2.11, this has a solution $\overline{\mathscr{A}}^{\mathrm{bg}}$ which we denote by $\mathscr{A}$.

Proposition 2.13. *The solution $\mathscr{A}$ to $\mathscr{A}^{bg}$ is the Néron model of A.*

Sketch of proof. Let $s \colon \mathscr{Z} \to \operatorname{Spec} R$ be a smooth R-scheme and fix a K-morphism $f_K \colon Z_K \to A$. Consider also the K-morphism $\tau_K \colon Z_K \times A \to A$ defined by $(z, x) \mapsto f_K(z)x$. By the WNMP of $\mathscr{A}^{\mathrm{wk}}$, this extends to a R-rational map $\tau^{\mathrm{wk}} \colon \mathscr{Z} \times \mathscr{A}^{\mathrm{wk}} \dashrightarrow \mathscr{A}^{\mathrm{wk}}$. Repeating the argument in the proof of Proposition 2.12, we get that the induced R-rational map $\mathscr{Z} \times \mathscr{A}^{\mathrm{wk}} \dashrightarrow \mathscr{Z} \times \mathscr{A}^{\mathrm{wk}}$ defined by $(z, x) \mapsto (z, \tau^{\mathrm{wk}}(z, x))$ restricts to a R-rational map $\mathscr{Z} \times \mathscr{A}^{\mathrm{bg}} \dashrightarrow \mathscr{Z} \times \mathscr{A}^{\mathrm{bg}}$. Since, by definition, the solution is birational to $\mathscr{A}^{\mathrm{bg}}$, then we view the latter as a R-rational map $\mathscr{Z} \times \overline{\mathscr{A}}^{\mathrm{bg}} \dashrightarrow \mathscr{Z} \times \overline{\mathscr{A}}^{\mathrm{bg}}$. Composing with the projection onto the second factor, we get the R-rational map $\bar{\tau} \colon \mathscr{Z} \times \overline{\mathscr{A}}^{\mathrm{bg}} \dashrightarrow \overline{\mathscr{A}}^{\mathrm{bg}}$ which extends τ_K. Since it is defined on the generic fibers, it has codimension 1. Thus by Weil Extension lemma it is defined everywhere and it extends to a morphism. Denote by $\varepsilon \colon \operatorname{Spec} R \to \mathscr{A}$ the unit section of $\mathscr{A}$, i.e., a section of the unique morphism $\mathscr{A} \to \operatorname{Spec} R$. Then the R-morphism

$$\bar{\tau} \circ (id, \varepsilon \circ s) \colon \mathscr{Z} \to \mathscr{Z} \times_R \mathscr{A} \to \mathscr{A}$$

extends $z \mapsto (z, 1_A) \mapsto f(z)1_A = f(z)$ and it coincides with f_K on the generic fiber. By the separatedness of $\mathscr{A}$ we have the unicity of the extension. Let me explain this last step in a more detailed way. In general, give two morphisms $f, g \colon \mathscr{X} \to \mathscr{Y}$ of R-schemes, one can define the locus where they agree as follows. This is a $\mathscr{X}$-scheme $\ell \colon \mathscr{L} \to \mathscr{X}$ defined by the following universal property. For any morphism $h \colon \mathscr{Z} \to \mathscr{X}$ we have $h \circ f = h \circ g$ if and only if h factors through ℓ. Such a scheme T exists, since it is represented by the fiber product

$$\begin{array}{ccc} \mathscr{L} & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \\ \mathscr{Y} & \xrightarrow{\ \Delta\ } & \mathscr{Y} \times_R \mathscr{Y} \end{array}$$

where Δ is the diagonal morphism. In particular, the separatedness of $\mathscr{A}$ implies that Δ is a closed embedding, hence so is ℓ. Therefore the locus where f and g agree is a closed subscheme of $\mathscr{X}$. In our context, since the morphisms agree on the generic fiber $\mathscr{Z}_K$, the locus of agreement is a closed subscheme of $\mathscr{Z}$ containing the generic fiber. Therefore it has to be everything, and so the two morphisms agree. □

2.2.5. *The global case.* An immediate goal after Theorem 2.4 consists of extending it to non-local Dedekind domains. As one may guess, this basically happens by glueing local Néron models together. Unfortunately, this global model may not be of finite type anymore, but in the case of abelian varieties the glueing works properly, and we obtain the desired global Néron model. Next lemma makes a more precise sense of the what a local Néron model is.

Lemma 2.14. *Let X be a smooth, separated K-scheme of finite type. Then:*

(1) *let $\mathscr{X}$ be a S-scheme of finite type with generic fiber X. The $\mathscr{X}$ is a Néron model of X over R if and only if the $\mathscr{O}_{S,s}$-scheme $\mathscr{X}_{(s)} = \mathscr{X} \times_S \mathscr{O}_{S,s}$ is a Néron model of X over $\mathscr{O}_{S,s}$ for every closed point $s \in S$;*
(2) *there exists a Néron model $\mathscr{X}$ over $S = \operatorname{Spec} R$ of X if and only if there exists a dense open subscheme $S' \subset S$, a Néron model $\mathscr{X}'$ over S' and Néron models of $\mathscr{X}_{(s)}$ of X over $\mathscr{O}_{S,s}$ for each of the finitely many $s \in S - S'$.*

Proof. (1) Let K denote the fractions field of $\mathscr{O}_{S,s}$. We want to show that $\mathscr{X}_{(s)}$ satisfies the NMP. Let $\mathscr{Y}_{(s)}$ be a smooth $\mathscr{O}_{S,s}$-scheme and pick a morphism $f_K \colon \mathscr{Y}_{(s)K} \to \mathscr{X}_{(s)K}$. Assume it is also of finite type, hence of finite presentation over $\mathscr{O}_{S,s}$. Then we can extend $\mathscr{Y}_{(s)}$ to a scheme $\mathscr{Y}'$ over a connected open neighborhood $S' \subset S$ of s and suppose it is smooth. Since $\mathscr{X}' = \mathscr{X} \times_S S'$ is a Néron model of its generic fiber, then it follows that f_K extends uniquely to a S'-morphism $f' \colon \mathscr{Y}' \to \mathscr{X}'$. Therefore $f' \otimes \mathscr{O}_{S,s}$ is the desired morphism.

On the other hand, consider a K-morphism $f_K \colon \mathscr{Y}_K \to \mathscr{X}_K$. As above, assume $\mathscr{Y}$ is of finite presentation over S. Then by the second part of lemma 2.5 we have that over a neighborhood U_s of a closed point $s \in S$, the morphism f_K extends uniquely to a U_s-morphism $f_s \colon \mathscr{Y} \times_S U_s \to \mathscr{X} \times_S U_s$. Glueing all the f_s's yields a unique S-morphism $f \colon \mathscr{Y} \to \mathscr{X}$ extending f_K. Finally, since $\mathscr{X} \times_S \operatorname{Spec} \mathscr{O}_{S,s}$ is smooth and separated, so is $\mathscr{X}$.

(2) We sketch the proof of (2). First of all, we claim that for an open covering $(S_i)_i$ of S, then $\mathscr{X}$ is a Néron model of X if and only if it is a Néron model for the S_i-scheme $X \times_S S_i$. This fact, combined with part (1) give the "if" part.

On the other hand, assume S is connected and denote the closed points of $S - S'$ by $\{s_1, \dots, s_n\}$. Let $\mathscr{X}'$ be a Néron model of X over S', and let $\mathscr{X}_{(s_i)}$ be the local Néron model of X over $\mathscr{O}_{S,s_i}$. Then by lemma 2.5 we have that $\mathscr{X}_{(s_i)}$ extends to a smooth, separated scheme $\mathscr{X}_i$ of finite type over U_i, an open neighborhood of s_i. Since $\mathscr{X}'$ and $\mathscr{X}_i$ coincide on the generic fiber, i.e., on the generic point of S, they also coincide on a non-empty part of S'. After removing finitely many points from U_i, assume that $U_i \cap (S - S') = \{s_i\}$ and that $\mathscr{X}_i$ agrees with $\mathscr{X}'$ on $S' \cap S_i$. Glueing the $\mathscr{X}_i$'s with $\mathscr{X}'$ over $S' \cap S_i$ gives us a smooth, separated S-model $\mathscr{X}$ of finite type such that $\mathscr{X} \times_S S' = \mathscr{X}'$ and $\mathscr{X} \times_S \operatorname{Spec} \mathscr{O}_{S,s_i} = \mathscr{X}_{(s_i)}$. Therefore $\mathscr{X}$ is a global Néron model of X by the first part of this lemma.

□

Lemma 2.15. *Let R be a Dedekind domain with fractions field K,a dn let A be an abelian variety over K. Assume that A extends to a smooth and proper R-scheme $\mathscr{A}$. Then $\mathscr{A}$ is an abelian scheme and its group structure extends the one on A.*

Proof. See [2, Proposition 2 p. 19]. □

Theorem 2.16. *Let R be a Dedekind domain with K its fractions field. Any abelian variety A over K has a global Néron model $\mathscr{A}$ over R.*

Sketch of proof. Lemma 2.5 grants us that we can "spread out" A to a scheme $\mathscr{A}^U$ of finite type over a neighborhood of the generic point of Spec R. By lemma 2.15 that $\mathscr{A}^U$ is smooth, proper and that admits a R-group law birational to the group law on A. This implies that $\mathscr{A}^U$ is a Néron model of A over U. Since by Theorem 2.4 there are Néron models over the local rings of the finitely many points in $\operatorname{Spec} R - U$, we conclude by Lemma 2.14. □

2.3. **The case of abelian schemes.** Despite the fact that Néron models are defined in the broad setting of schemes, it seems that their major use lies in the realm of group schemes. The classical non-example of a Néron model consists of considering $\mathbb{P}^1_K$ and its smooth, separated R-model $\mathbb{P}^1_R$, which fails to be its Néron model. In fact, not all automorphisms of $\mathbb{P}^1_K$ extend to those of $\mathbb{P}^1_R$, as there are matrices in $\mathrm{GL}_2(K)$ that do not remain invertible once rescaled so to have coefficients in R.

Note that not every group scheme has a Néron model (or an obvious one): for instance $\mathbb{A}^1_R$ is not the Néron model of $\mathbb{A}^1_K$, since $\mathbb{A}^1_K(K) \neq \mathbb{A}^1_R(R)$. Actually, $\mathbb{A}^1_K$ does not have any Néron model. Similarly, $\mathbb{G}_{m,R}$ is not the Néron model of $\mathbb{G}_{m,K}$, although $\mathbb{G}_{m,K}$ has a locally of finite type Néron model, which we are going to construct. One trivial remark, which amounts to non-trivial hint: one can write $\mathbb{Q}_p^\times = \bigsqcup_{n\in\mathbb{Z}} p^n\mathbb{Z}_p^\times$.

2.3.1. *The Néron model of the multiplicative group and of the norm torus.* Let us consider the case of a dvr R with uniformizer ϖ and fraction field K. Let $S = \operatorname{Spec}(R)$ and let s be the closed point of S. Indeed, ϖ is the generator of the ideal corresponding to s over an open dense (i.e., non-empty) neighborhood $U(s)$ of s. Note that, away from s, the uniformizer ϖ is invertible, since it generates the ideal corresponding to s. Hence, for $n \in \mathbb{Z}$, we can (and will) consider ϖ^n as a $(U(s) - \{s\})$-valued point of $\mathbb{G}_{m,R}$. Consider now

$$\varpi^n\mathbb{G}_{m,R}$$

as the translate by ϖ^n of $\mathbb{G}_{m,R}$ inside the R-model we want to construct. Note that, as schemes, $\varpi^n\mathbb{G}_{m,R}$ and $\mathbb{G}_{m,R}$ are isomorphic. The point is that, over the generic fiber, the way it is glued is different. The idea is that we have $\varpi^n\mathbb{G}_{m,R} \simeq \operatorname{Spec} R[X, X^{-1}]$, but the point on the generic fiber corresponding to an element $r \in R$ of valuation n comes from the morphism

$$R[X, X^{-1}] \to R, \quad X \mapsto \frac{r}{\varpi^n}$$

since now r/ϖ^n is a unit over R. In other words, we have the glueing data over the generic fiber K

$$\varphi_{n,m} \colon \varpi^n\mathbb{G}_{m,R} \times \operatorname{Spec} K \to \varpi^m\mathbb{G}_{m,R} \times \operatorname{Spec} K$$

where $\varphi_{n,m} = \cdot\varpi^{m-n}$, which is an isomorphism over K. In this way we obtain the equivalence relation $\sim$ such that, for $x_n \in \varpi^n\mathbb{G}_{m,R}$ and $x_m \in \varpi^m\mathbb{G}_{m,R}$, we have $x_n \sim x_m$ if and only if $\varphi_{n,m}x_n = x_m$.

We thus define

$$\mathscr{G}_m = \bigsqcup_{n\in\mathbb{Z}} \varpi^n\mathbb{G}_{m,R}/\sim .$$

This is a smooth, separated R-group scheme *locally* of finite type. Since ϖ is invertible in K, the glueing data are isomorphism over the generic fiber and therefore $\mathscr{G}_m \times \operatorname{Spec}(K) \simeq \mathbb{G}_{m,K}$. On the other hand we have

$$\mathscr{G}_m(R) = \bigsqcup_{n} \varpi^n\mathbb{G}_{m,R}(R) = K^\times,$$

and by [2, Proposition 2, p.290] we conclude that $\mathscr{G}_m$ is the lft Néron R-model of $\mathbb{G}_{m,K}$.

Let us now consider the so called *norm torus*, i.e., the affine scheme given by a Pell equation. Consider $\mathbb{Z}_p$ for p odd. We consider the following $\mathbb{Q}_p$-norm torus

$$T = \operatorname{Spec} \frac{\mathbb{Q}_p[X,Y]}{(X^2 - pY^2 - 1)}.$$

This is a $\mathbb{Q}_p$-group scheme with multiplication, inverse, and unit section respectively given by:

$$(a, b) \cdot (a', b') = (aa' + pbb', ab' + a'b),\ (1, 0),\ (a, b)^{-1} = (a, -b).$$

This $\mathbb{Q}_p$-norm torus happens to have a nice Néron model: we show that the smooth, separated, $\mathbb{Z}_p$-group scheme

$$\mathscr{T} = \operatorname{Spec} \frac{\mathbb{Z}_p[X,Y]}{(X^2 - pY^2 - 1)}$$

of finite type is in fact the Néron model of T. In fact, it is enough to check that its Q_p^{ur}-points are in bijection to its $\mathbb{Z}_p^{\mathrm{ur}}$, where the superscript ur denotes the maximal unramified extension. Consider a solution $(a/p^n, b/p^m)$, for p not dividing a and b (so that the solution is not integral) and $n > m$. Suppose, by contradiction, than $n > 0$. Then $p^{2n} = a^2 - p^{1-m+2n}b^2$. Reducing mod$p$, this implies that p divides a, which is a contradiction. Therefore $n = 0$. A similar argument for m allows us to conclude.

At any rate, note that there exist rational solution to the Pell equation that are *not* integral.

We conclude by noting that we have

$$T \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\sqrt{p}) \simeq \mathbb{G}_{m,\mathbb{Q}_p(\sqrt{p})}$$

as $\mathbb{Q}_p(\sqrt{p})$-schemes. This gives an exemples of how the formation of Néron models does not commute with ramified base change.

2.3.2. *Abelian schemes.* Recall that an abelian R-scheme $\mathscr{A}$, is a smooth, proper R-group scheme with connected geometric fibers. We remark that the commutativity is automatic, although it relies on some GIT deformation-theoretic arguments.

Proposition 2.17. *Let $\mathscr{A}$ be an abelian R-scheme. Then $\mathscr{A}$ is the Néron model of its generic fiber A.*

Proof. Let $\mathscr{Y}$ be a smooth R-scheme with generic fiber Y and a K-morphism $f_K \colon Y \to A$. We want to show that there exists a unique R-morphism $f \colon \mathscr{Y} \to \mathscr{A}$ extending f_K.

As $\mathscr{A}$ is separated, we can work locally on $\mathscr{Y}$. Assume, by smoothness, that $\mathscr{Y}$ is irreducible. By Lemma 2.5, the map f_K gives rise to a morphism $f' \colon \mathscr{Y} \times_R U' \to \mathscr{A}$ defined over an open neighborhood U' of the generic point of $\operatorname{Spec} R$.

Now we aim to extend f' to a R-rational map $f \colon \mathscr{Y} \dashrightarrow \mathscr{A}$. Let $s \in \operatorname{Spec}(R) - U$ be a closed point and consider the generic point η of an irreducible component $\mathscr{Y}_{s,i}$ of $\mathscr{Y}_s$. Let me recall now that R, as a dvr, can be thought of as a local ring of a codimension-1 closed set. Since $\mathscr{Y}$ is necessarily flat over R, we have the formula $\dim \mathscr{Y} - \dim \mathscr{Y}_s = \dim R$, so the the fiber of $\mathscr{Y}$ at the closed point s is of 1-codimensional. Namely, say that $\dim Y = n$. Then we can view $\mathscr{Y}$ as a family of n-dimensional fibers over the 1-dimensional base $\operatorname{Spec} R$, so that it is $(n+1)$-dimensional. Hence its fibers are 1-codimensional in $\mathscr{Y}$. This implies that its local ring $\mathscr{O}_{\mathscr{Y},\eta}$ is a dvr. The K-morphism f_K induces the diagram

$$\begin{array}{ccccccc}
\operatorname{Spec}(Frac(\mathscr{O}_{\mathscr{Y},\eta})) & \longrightarrow & Y & \xrightarrow{f_K} & A & \longrightarrow & \mathscr{A} \\
\downarrow & & & \exists! & & & \downarrow \\
\operatorname{Spec} \mathscr{O}_{\mathscr{Y},\eta} & & & \longrightarrow & & & \operatorname{Spec} R
\end{array}$$

and by the valuative criterion for properness for $\mathscr{A}$, there exists a unique such dotted map. In other words, f_K extends uniquely to a morphism $\operatorname{Spec} \mathscr{O}_{\mathscr{Y},\eta} \to \mathscr{A}$. By Lemma 2.5, we obtain an extension of f_K to a morphism $U_\eta \to \mathscr{A}$ for U an open neighborhood of η. Since $\mathscr{A}$ is separated, we can glue together $U_\eta \to \mathscr{A}$ with f' and obtain a map $f \colon U' \cup U_\eta \to \mathscr{A}$ which extends f_K. Furthermore we have that $U' \cup U_\eta$ is S-dense because of the generic points. Hence $f \colon \mathscr{Y} \dashrightarrow \mathscr{A}$ is an S-rational map extending f_K. Note that by construction $U' \cup U_\eta$, we have that $\operatorname{dom}(f)$ contains all 1-codimensional points of $\mathscr{Y}$ in the closed fibers. Since $\mathscr{Y}$ is a smooth scheme and $\mathscr{A}$ is smooth and proper, hence separated, group scheme, by Weil Extension lemma we obtain that f is defined everywhere. □

2.3.3. *Elliptic curves.* Consider now an elliptic curve E over K and let $\mathscr{E}$ be its Néron model. A nice model to deal with, especially for explicit calculations, is the minimal Weierstrass model $\mathcal{W}$, whenever it exists[7]. For E with good reduction, $\mathcal{W}$ is an abelian scheme, so by Proposition 2.17 we have that $\mathcal{W} \simeq \mathscr{E}$.

[7]In our setting, for R a dvr, it does.

On the other hand, in the bad reduction case, $\mathcal{W}$ is not smooth.
Let $\mathcal{E}$ the minimal regular proper model of E, i.e., a regular, proper R-model of E which is minimal with respect to the relation of domination among all other regular R-models of E. Namely, for another such a model $\mathcal{E}'$ of E, any domination map $\mathcal{E} \to \mathcal{E}'$ is an isomorphism. Intuitively, one may think of it as of the regular proper R-model such that the special fiber has the lowest possible number of components, and blowing down one of them would imply a loss of regularity. One can find a regular model by repeatedly blowing-up and normalizing a model. Subsequently, one finds a minimal regular model by blowing-down certain divisors in the special fiber.
It is known[8] that for any smooth K-curve of non-zero genus, there exists a unique minimal regular proper model. Let $\mathcal{E}^{\mathrm{sm}}$ be the smooth locus of $\mathcal{E}$. By [2, Prop.1, p.21] we have that the map induced by the NMP of $\mathscr{E}$ gives the following isomorphism

$$\mathcal{E}^{\mathrm{sm}} \simeq \mathscr{E}. \tag{2.1}$$

Next result aims to a rather "concrete" description of $\mathscr{E}$. Let us denote by $\mathscr{E}^{\circ}$ the relative identity component, i.e., the open subscheme of $\mathscr{E}$ given by removing the closed complement of the identity component in the (finitely many) disconnected fibers. Then we have

$$\mathcal{W}^{\mathrm{sm}} \simeq \mathscr{E}^{\circ},$$

which essentially follows from (2.1) and by the fact that a minimal Weierstrass model can be obtained by blowing down the (finitely many) components of the special fiber $\mathcal{E} \times_R \operatorname{Spec} k$ which are disjoint from the closure in $\mathcal{E}$ of the identity of E.

Here the sketch of an example. Let E be defined by $y^2 = x^3 + p$ over $\mathbb{Q}_p$. The same equation defines a minimal Weierstrass model $\mathcal{W}$ over $\mathbb{Z}_p$, which is smooth away from the point $(0,0)$ in the special fiber. Note that $(0,0)$ is regular. Thus $\mathcal{W}$ is a regular model of E, and it is also minimal since the special fiber has no divisor to blow down. Therefore we have that $\mathcal{W} - \{(0,0)\} \simeq \mathscr{E}$.

More generally, the following result [6, Theorem 4.1] holds.

Proposition 2.18. *Let C be a proper, smooth, connected curve of positive genus over K. Then the smooth locus $\mathscr{C}^{sm}$ of the minimal proper regular model of C over S is the Néron model of C.*

2.3.4. *Semi-stable reduction.* Recall that an abelian variety A over K has *semi-stable reduction* if the special fiber of the identity component of its Néron model is semi-abelian.

Theorem 2.19 (Semi-stable reduction)**.** *For an abelian variety A over K, there exists a finite separable extension K' over K such that $A \otimes_K K'$ has semi-stable reduction over the integral closure R' of R in K.*

Theorem 2.20. *Let $\mathscr{M}$ be a semi-abelian R-scheme with generic fiber the abelian variety A, whose Néron model is denoted by $\mathscr{A}$. Then the natural map*

$$\mathscr{M} \to \mathscr{A}^{\circ}$$

is an isomorphism.

As an immediate corollary to the previous theorem, one obtains that base-change behaves well in the semi-stable case: more precisely, the identity component is preserved under base-change.

Corollary 2.21. *Let $\mathscr{N}$ denote the Néron model of $A \otimes_K K'$. If A has semi-stable reduction, then the natural map*

$$\mathscr{A}^{\circ} \times \operatorname{Spec} R' \to \mathscr{N}^{\circ}$$

is an isomorphism.

[8]By an highly non-trivial theorem.

Proof. Since the $\mathscr{A} \times \operatorname{Spec} R'$ is smooth and separated with generic fiber $A_{K'}$, the by the NMP there is a unique R'-group map $\mathscr{A} \times \operatorname{Spec} R' \to \mathscr{N}$, which is the base-change morphism for a Néron model relative to $R' \to R$. This induces a natural map between the identity components. Now, we have that $\mathscr{A}^\circ \times \operatorname{Spec} R'$ is a semi-abelian R'-scheme with generic fiber $A_{K'}$. Thus by theorem 2.20 we conclude. □

2.3.5. *Néron models of Jacobians.* Recall that the Jacobian of a normal, flat K-curve C is a smooth, connected K-group scheme of finite type. If K is a perfect field, then normal is equivalent to smooth and we have that the Jacobian of a proper and smooth curve is an abelian variety, and so we know it has a Néron model. Next result deals with the case of a general base field K.
Quite naturally, one could object: why don we care about imperfect fields? Let $\mathscr{S}$ be a surface over $\mathbb{Z}$ and let η be a generic point of its special fiber at p. Then the local ring $\mathscr{O}_{\mathscr{S}_p,\eta}$ is a dvr whose residue field is global function fields over finite field, i.e., a function field of irreducible components of the fiber at p, and this global function field is imperfect.

Theorem 2.22. *Let $\mathscr{C}$ be a flat, regular projective R-curve with geometrically reduced and irreducible fibers, and assume its generic fiber C admits a section. Then* $\operatorname{Pic}^\circ{}_{\mathscr{C}/R}$ *is the Néron model of its generic fiber* $\operatorname{Pic}^\circ{}_{C/K}$.

Sketch of proof. Let $\mathscr{Y}$ be a smooth R-scheme and pick a K-morphism $f_K\colon \mathscr{Y}_K \to \operatorname{Pic}_{C/K}$.

Let $\operatorname{Pic}(C)$ denote the absolute Picard group, i.e., $H^1(C, \mathscr{O}_C^\times)$. Since C admits a section, by [2, Proposition 4, p.204], the sequence

$$\operatorname{Pic}(\mathscr{Y}) \to \operatorname{Pic}(\mathscr{C} \times_R \mathscr{Y}) \to \operatorname{Pic}_{\mathscr{C}/R}(\mathscr{Y})$$

is exact. Hence f_K corresponds to a line bundle L_K on $\mathscr{C}_K \times_K \mathscr{Y}_K$. Since by [2, Proposition 9, p.49] smooth schemes over regular schemes are regular, then $\mathscr{C} \times_R \mathscr{Y}$ is regular and L_K extends to a line bundle L on $\mathscr{C} \times_R \mathscr{Y}$. Therefore f_k extends to a R-morphism $f\colon \mathscr{Y} \to \operatorname{Pic}_{\mathscr{C}/R}$.

By constancy of the degree map in flat families (see [2, Proposition 2, p.238]), the map f factors through $\operatorname{Pic}^\circ{}_{\mathscr{C}/R}$. The unicity follows from the separatedness of $\operatorname{Pic}^\circ{}_{\mathscr{C}/R}$.

□

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